Simulation Infiltration Rate of Water on Sand Media by Finite Difference Method

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Abstract. Simulation Infiltration Rate of water on two dimension of thin plate soil by using finite difference method. The water flow in unsaturated zone explained by Nonlinear Richard equation mathematically. Richard equation is a partial differential equation, parabolic type. The finite difference approximate that recommended to solve this equation is ADI (Alternating Direction Implicit) method. The distribution of water content from finite difference method calculation is compared to the distribution of water content from analytic calculation. The error percentage of model is depending on size of mesh grid, they are 4.0 cm, 2.0 cm and 1.0 cm and the errors in a row are 0.31%, 0.43% and 0.52%. The type of soil are interpreted in this simulation is sand. The simulation show that the longer time duration the smaller infiltration rate on the sand field. **Keywords**: *Infiltration Rate, ADI Method, Campbell Relationship*

INTRODUCTION

The water flow from the surface into the soil is called infiltration. This flow is an unsteady flow and occurs on unsaturated zone of soil. Hydraulic phenomena have been studied for the past fifteen years, developing in this manner numerical and mathematical tools in order to simulate the widest possible range of mixed flows, from overland flow (using diffusive wave approximate models) to the river networks [13]. In order to simulate the unsteady hydraulic phenomena so it's needed a general equation of water infiltration.

The general equation of infiltration for the flow on unsteady zone formulated by combining the relationship between conservative mass and the Darcy's law. This combination results a partial differential equation called Richard equation. From the Richard equation the water infiltration is modelled, by using numerical approximation. Numerical method that usually used such as finite difference method, finite element method etc.

Finite difference method is one of methods that usually implemented to explain the physics phenomena because this method has ability to solve the partial differential equation with a solution that approaches the exact value. The finite difference method proceeds by replacing the derivatives in the differential equations by finite difference approximations. This gives a large algebraic system of equation to be solved in place of the differential equation, something that easily solved on a computer [11]. Whereas the finite element method is appropriate with the complex geometric [15], for example Tomography problem, resistivity (Electrical) and fluid network. Each method has advantages and disadvantages.

The Richard equation is a partial differential equation, parabolic type. There are two finite difference approximations that can be used to solve this equation, which is an implicit and explicit method. The part of explicit method is forward time central space (FTCS) method and DuFort-Frankel method. Whereas the part of *implicit* method is *Crank Nicolson* method dan *Back time central space* (BTCS) method etc.

Finite difference method with the highest recommendation to solve parabolic equation is Crank Nicolson method (for one dimension problem) [4]. However, on two dimension problems, *Crank Nicolson* is unconditionally stable, but it requires more operations per time step than the number of unknown variables [10] along with the large matrix which is no longer tridiagonal.

One way to overcome the shortcomings and inefficiencies of the Crank Nicolson method is to use a splitting method. This method is known as the alternating direction implisit method or ADI [9]. According to Gurevich (2008) ADI method for two dimensions has an absolute stable condition. Thus, this method is precise for solving two dimensions of Richard equation.

The simulation is built by ADI method using a centered finite difference, this means that the condition of one node is influenced by other nodes around (neighbor nodes) for each time step. On this scheme occurs alteration which is very fast and have a lot of calculations, therefore computer assistance is needed.

CONTENT AND METHOD

Water flow from the surface in to the soil in nature, usually occur on unsaturated zone. The unsaturated zone plays an important role in the hydrological cycle. Its forms the link between surface water and ground water and has dominant influence on the partitioning of water between them [7].

The general equation for unsaturated flow is derived by combining the conservation of mass (S) with Darcy's law. If we assume constant fluid density [6], so:

$$S_s = \frac{\partial\theta}{\partial h} = -\frac{\partial\theta}{\partial h}\frac{\partial h}{\partial t} = \frac{\partial\theta}{\partial t} \tag{1}$$

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} = S_s = -\frac{\partial\theta}{\partial t} \tag{2}$$

Now substituting Darcy's law for unsaturated flow for each coordinate direction into equation (2) then

$$\frac{\partial}{\partial x}\left[K(\theta)\frac{\partial h(\theta)}{\partial x}\right] + \frac{\partial}{\partial y}\left[K(\theta)\frac{\partial h(\theta)}{\partial y}\right] + \frac{\partial}{\partial z}\left[K(\theta)\frac{\partial(h(\theta)+1)}{\partial z}\right] = \frac{\partial\theta}{\partial t} \tag{3}$$

$D(\theta)$ is the hydraulic diffusivity (m/s^2) and it's different for wetting and drying [7]. $D(\theta)$ is defined as

$$D(\theta) = K(\theta).\frac{\partial h(\theta)}{\partial\theta} \tag{4}$$

Diffusivity can be expressed as an analytical function of soil properties and water content using the Campbell's relations

$$D(\theta) = -b.\psi_{ae}K_h.\phi^{-b-3}.\theta^{b+2} \tag{5}$$

Where θ is the water content in the pores of sand, ϕ is porosity, ψ_{ae} is the air –entry pressure head, and K_h is the saturated hydraulic conductivity. Thus $D(\theta) > 0$ ($\psi_{ae} < 0$) and increases with water content [5].

When the fluid flowing, the movement of water occur on three spatial direction, and for decreasing the complicated of mathematic then considered that the flow just occur on two dimension [2]. Thus equation (3) can be change to two dimension form:

$$\frac{\partial}{\partial x}\left[K(\theta)\frac{\partial h(\theta)}{\partial x}\right]+\frac{\partial}{\partial y}\left[K(\theta)\frac{\partial h(\theta)}{\partial y}\right]=\frac{\partial \theta}{\partial t} \tag{6}$$

According to equation (4), then equation (6) is written as:

$$\frac{\partial}{\partial x}\left[D(\theta)\frac{\partial \theta}{\partial x}\right]+\frac{\partial}{\partial y}\left[D(\theta)\frac{\partial \theta}{\partial y}\right]=\frac{\partial \theta}{\partial t} \tag{7}$$

With assuming that the field of soil is homogeny and isotropic, equation (8) has written as:

$$D(\theta)\left[\frac{\partial^2 \theta}{\partial x^2}+\frac{\partial^2 \theta}{\partial y^2}\right]=\frac{\partial \theta}{\partial t} \tag{8}$$

Equation (8) gets by assume that hydraulic diffusivity $D(\theta)$ not change toward the spatial variables, but it's a function of water content before (base on equation (5)).

By using ADI method equation (8) will be solve. For the first step, fix the calculation on the y –direction, Use equation (9).

$$D(\theta)\left[\frac{\theta_{i+1,j}^{t}-2\theta_{i,j}^{t}+\theta_{i-1,j}^{t}}{(\Delta x)^2}+\frac{\theta_{i,j+1}^{t+\frac{1}{2}}-2\theta_{i,j}^{t+\frac{1}{2}}+\theta_{i,j-1}^{t+\frac{1}{2}}}{(\Delta y)^2}\right]=\frac{\theta_{i,j}^{t+1/2}-\theta_{i,j}^{t}}{\Delta t/2} \tag{9}$$

Then fix the calculation on the x –direction for the second step, by use equation (10).

$$D(\theta)\left[\frac{\theta_{i+1,j}^{t+1}-2\theta_{i,j}^{t+1}+\theta_{i-1,j}^{t+1}}{(\Delta x)^2}+\frac{\theta_{i,j+1}^{t+\frac{1}{2}}-2\theta_{i,j}^{t+\frac{1}{2}}+\theta_{i,j-1}^{t+\frac{1}{2}}}{(\Delta y)^2}\right]=\frac{\theta_{i,j}^{t+1}-\theta_{i,j}^{t+1/2}}{\Delta t/2} \tag{10}$$

Equation (9) and (10), is a solution of Richard equation, by using ADI method. $D_x = D_y = D(\theta)$, the stable condition for ADI method explained with equation (11).

$$2D(\theta) \leq \frac{1}{2} \tag{11}$$

Both of heat transfer and infiltration problem, is a part of transient problem, known as transient flow. According to Bhattacharjya (2016), analytical solution for diffusion equation can get with this way. Diffusion equation for homogeny media is written as:

$$D_x\frac{\partial^2 C}{\partial x^2}+D_y\frac{\partial^2 C}{\partial y^2}=\frac{\partial C}{\partial t} \tag{12}$$

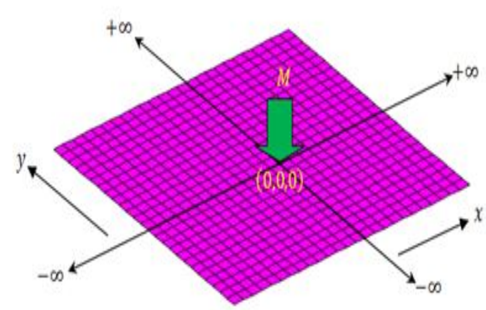

FIGURE 1: Loading of solute of mass M in a 2D unbounded system

Consider the 2D unbounded domain (Figure 1). The solution of the diffusion equation will give us the spatial and temporal distribution concentration in the unbounded domain.

$$C(x=0, y=0, t=0) = M\delta(x, y) \tag{13}$$

Where δ is the kronecker delta. $\delta(x, y) = 0$, for x, and $y \neq 0$, $\delta(x) = 1$, for $x, y = 0$. Boundary conditions,

When $x \to \pm\infty$ $\quad \partial C/\partial x, C \to 0$

When $y \to \pm\infty$ $\quad \partial C/\partial y, C \to 0$

The diffusion equation (12), for the initial and boundary condition can be solved using separation of variable method.

$$C(x, y, t) = C_1(x, t)C_2(y, t) \tag{14}$$

And the solution for equation (14) is,

$$C(x, y, t) = \left(\frac{M}{4\pi t\sqrt{D_x D_y}}\right) e^{-\left(\frac{x^2}{4D_x t}+\frac{y^2}{4D_y t}\right)} \tag{15}$$

Equation (15) is analytical solution of diffusion equation that can be used for distribution of water in the soil pores by using *M* value as the input rate cross the area.

Geometric Models and Modelling

Model that used in this simulation is two dimension field of sand. On this model, water distribution will observe at every nodes on the field with size 60 cm × 60 cm. There are three variation of grid size used in this model 4.0 cm, 2.0 cm, and 1.0 cm. This test to purpose that the grid size have influence on the error percentage of the model.

Simulation made base on the ponding system that usually used for measure the infiltration rate by using infiltration ring in the real system. In order to make the infiltration occur, regards that there are ponding on one of the field side. Then the water flow into the sand field will occur, laminar and continuously.

Besides the assumption above, modelling infiltration of water in sand media using another assumption. The field that used is homogeneous and isotropic without interruption to the system. The water flow entering from the side which axis y=n, along of the x –axis. The pores of the soil on this side is in saturation state (θ =θ_s). While the part of the other field is in dry state (θ =θ_{fc}).

At the time t=0, the initial percentage of water content in the sand field is 4.89%. After the increase of one unit of time, the new of water content will be quantify by summing the initial water content with the change of water content obtained by ADI method.

The equation that used to calculate the water content numerically are equation (9) and (10). While the equation to calculate the water content analytically is equation (15). The result (the new water content) will substituted into the function of hydraulic diffusivity to obtain the value of the diffusion of the new ones. The process will be repeated until it reaches a maximum predetermined time.

Initial and Boundary Condition

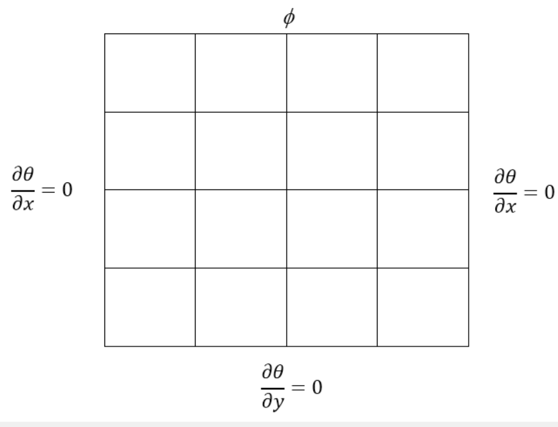

FIGURE 2. The boundary condition on the domain

According to Figure 2, the boundary condition of the model is homogeneous Neumann boundary condition on the bottom, left, and right side because on this limit there is no flow into the field. Conformity with the assumptions that have been made, the initial term of the model is:

$\theta = \phi, t \geq 0,$ when $x \geq, y = n$

$\theta = \theta_{fc}, t = 0,$ when $x \geq 0,\ 0 < y < n$

Where n is the limit of y –axis.

Validation of Model

Validation of model performed by numerical analysis with use equation (16). Then compare the results obtained with the analytical solution and the results obtained by using ADI method with use equation (17).

$$Error = \frac{\Sigma(\theta_{new} - \theta_{previous})^2}{number\ of\ nodes} \tag{16}$$

$$Error = \frac{\Sigma\left(\frac{Analytical\ results - Numerical\ results}{Analytical\ results}\right)}{Number\ of\ nodes} \times 100\% \tag{17}$$

RESULT AND DISCUSSION

This simulation is built by using Python 3.4.3 under windows 7 operating system. After the simulation, the result of simulation data is stored in PNG image file (.png) for picture data and in Python file (.py) for numeric data. Water content at unknown nodes calculated by the finite difference method for each increment of time. Then, the contour plots are processed against those nodes to obtain a picture of water distribution in the sand field.

Water Distribution in the Sand Field

Figure 3 shows the percentage of water in the field of sand at three different time duration. The initial percentage of water content is 4.89% ($\theta_{fc} = 0.0489$), and the characteristics of the sand according to Brook-Corey and Campbell parameters are $\theta_s = 0.451$, $\psi_{ae} = -2.20$ cm, b= 2.27 and coefficient of permeability (K_h) is 1,76 $\times 10^{-2}$cm/s . The contour lines are increasingly tenuous at any increase of time duration, which indicates that water content increases in the sand pores.

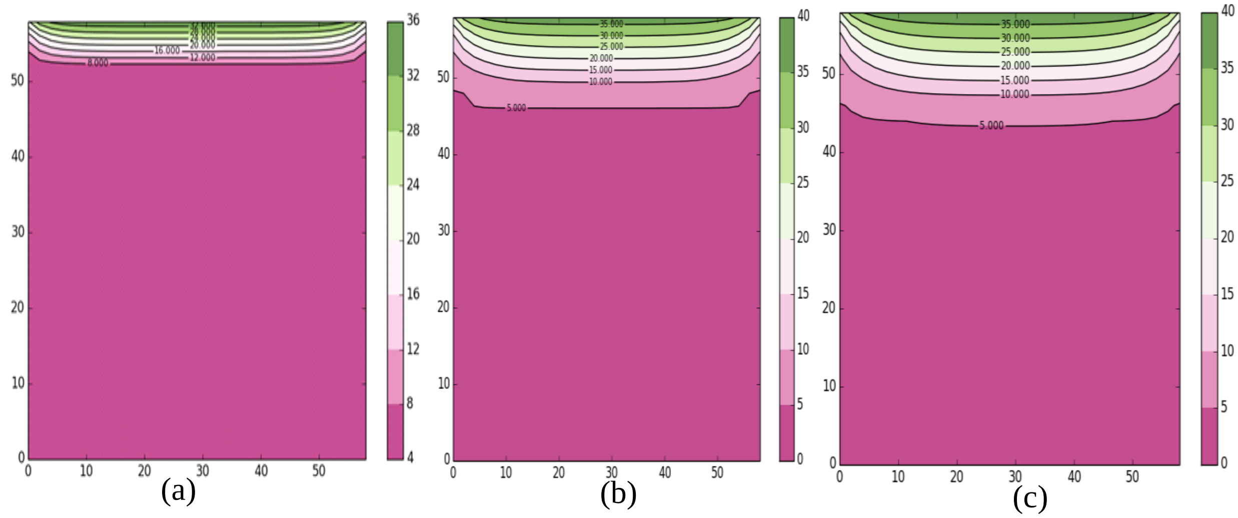

FIGURE 3. The percentage of water content on the sand field a) at time duration 86,400 units of time, b) at time duration 172,800 units of time, c) at time duration 259,200 units of time

Water Input Rate and Infiltration Rate

The volume of water that enters into the field marked by the changes of water content in the field. Thus, the water input rate is the change of water content in the field towards the increase of time. The average of water input rate obtained by averaging the changes of water content at all nodes. The graph of water input rate is shown in Figure 4.

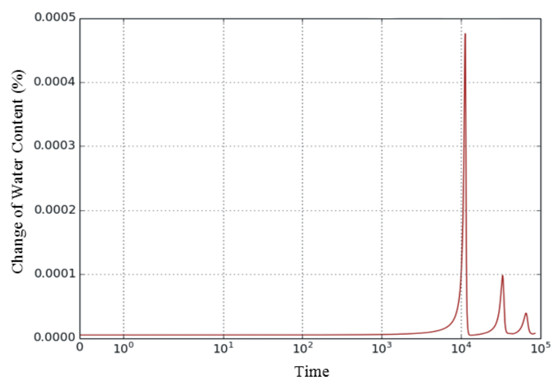

FIGURE 4. The graph of water input rate on the sand field

The infiltration of water in the sand field obtained from the simulation result shown in Table 1.

TABLE 1. Infiltration rate of water on the sand field

Units of Time	Rate (cm/Units of Time)
86,400	$\pm 9.26 \times 10^{-5}$
172,800	$\pm 6.94 \times 10^{-5}$
259,200	$\pm 3.47 \times 10^{-5}$

Infiltration rate in Table 1 is the rate of the y –axis. The infiltration rate is calculated only to see if the model has fulfilled the conditions of the infiltration in nature. Thus, the value of infiltration rate from this simulation different from infiltration in the real world because on this model, the gravitational force is negligible.

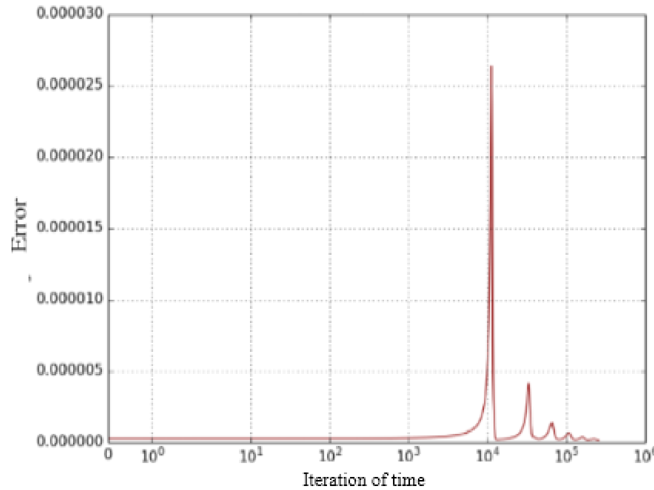

FIGURE 5. Relationship graph between the errors toward the iteration of time on the sand field

Figure 5 shows the distribution of error, with the fault tolerance gained is 10^{-6}. Data show that the converging condition achieved on literacy 69,052. On the chart error distribution, there is a growth of amplitude. This situation is classified as instability static.

The result of the comparison between the calculation by using analytical solutions and using ADI method can be seen on Table 2.

TABLE 2. Percentage of error due to variations grid size

$\Delta x = \Delta y$ (cm)	Error (%)
4.0	0.31
2.0	0.43
1.0	0.52

Based on Figure 3, shows an increase of water content in the pores of sand along the field of the increment of time. This identifies that a water distribution occurs in the sand field. The flow occurs due to the difference of pressure within the sand pores are full of water and the dry pores. Variations of time duration is in the great ranges, because the water movement in unsaturated zone is relative very slow. It is because the water must fill the sand pores in the dry state without the gravity.

Figure 4 shows the fluctuation graph. At the beginning of the process, the input rate of water is very small then it increases until it reaches the maximum rate. This happens due to the increase of hydraulic diffusivity in the sand field as the result of an exponential relationship between the hydraulic diffusivity and the water content. When the hydraulic diffusivity begins to reach steady state (*water content* ≈ *porosity*) then the input rate of the water will begin to decline and in the end it reaches zero value and the sand field is in a saturation state. However, the results that obtained by ADI method, giving a fluctuation graph. This fluctuation graph is due to the application of nonlinear Richard equation which is a combination of Conservative Law and Darcy's law. When the numeric scheme is applied on a conservative law, there will be spurious oscillations and resulting a weak solution [12].

Table 1 shows that the longer time duration the smaller infiltration rate on the sand field. The equilibrium of infiltration rate occurs at the time duration is 259,200 units time. Table 2 shows that the reduction size of the grid cannot reduce the percentage of error. In the unsaturated flow situation, soil hydraulic conductivity varies approximately exponentially with water content and may exhibit large differences over a flow section. For this reason, one cannot model a large section with a geometrically similar smaller one without making scaling adjustments in the relationship between hydraulic conductivity and pressure head [1].

It would be difficult to obtain the result that exactly approaches the analytical results. It's because the Richard equation is a nonlinear equation [6]. The error percentage of this model is depending on the size of mesh grid. They are 4.0 cm, 2.0 cm and 1.0 cm and the errors in a row are 0.31%, 0.43% and 0.52% which are quite small. Thus, the finite difference method can be applied in modelling the infiltration of water in the sand field.

CONCLUSION

The results of infiltration rate of water modelling with ADI method in sand media reaches the equilibrium state of infiltration rate at the time duration 259,200 units of time. The simulation shows that the longer time duration the smaller the infiltration rate. The error of the model depends on the size of the grid that used. They are 0.31% for the grid size 4.0 cm, 0.43% for the grid size of 2.0 cm and 0.52% for the grid size 1.0 cm. The error tolerance by numerical analysis is 10^{-6}.

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