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On The Boundedness Properties of the Generalized Fractional Integrals on the Generalized Weighted Morrey Spaces

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Abstrak. *Ruang Morrey pertama kali diperkenalkan oleh C.B. Morrey pada tahun 1938. Ruang Morrey dapat dipandang sebagai generalisasi dari ruang Lebesgue. Ruang Morrey kemudian diperumum menjadi ruang Morrey diperumum, ruang Morrey berbobot, dan ruang Morrey berbobot diperumum. Salah satu kajian pada ruang Morrey adalah mengenai keterbasan operator pada ruang tersebut. Salah satu operator tersebut adalah integral fraksional. Keterbatasan integral fraksional pada ruang Morrey klasik, ruang Morrey berbobot, ruang Morrey diperumum, dan ruang Morrey berbobot diperumum telah diketahui. Salah satu generalisasi dari integral fraksional adalah integral fraksional diperumum. Operator tersebut terbatas pada ruang Morrey diperumum. Tujuan penelitian ini adalah menyelidiki keterbatasan integral fraksional diperumum pada ruang Morrey berbobot diperumum. Bobot yang digunakan adalah Muckenhoupt class. Hasil yang diperoleh menunjukkan bahwa operator fraksional diperumum terbatas antar ruang Morrey berbobot diperumum dengan asumsi-asumsi tertentu. Hasil yang diperoleh kemudian mengimplikasikan keterbatasan dari integral fraksional maksimal diperumum pada ruang Morrey berbobot diperumum dengan asumsi-asumsi yang sama.*
Kata Kunci: *Integral fraksional diperumum, Muckenhoupt class, ruang Morrey berbobot diperumum.*

Abstrak. *Morrey Spaces were first introduced by C.B. Morrey in 1938. Morrey space can be considered as a generalization of the Lebesgue spaces. Morrey spaces were then generalized become the generalized Morrey spaces, the weighted Morrey spaces, and the generalized weighted Morrey spaces. One of the studies on Morrey spaces is the boundedness of certain operators on the spaces. One of the operators is the fractional integral. The boundedness of fractional integrals on the classical Morrey spaces, the weighted Morrey spaces, the generalized Morrey spaces, and the generalized weighted Morrey spaces had been known. One of the extensions of fractional integrals is generalized fractional integral. The operator was bounded on the generalized Morrey spaces. The purpose of this study is to investigate the boundedness of generalized fractional integrals on the generalized weighted Morrey spaces. The weight used is Muckenhoupt class. The results obtained show that the generalized fractional integral is bounded from generalized weighted Morrey space to another generalized weighted Morrey space under some assumptions. The main result obtained then implies the boundedness of the generalized fractional maximal operator on generalized weighted Morrey spaces under the same assumptions.*
Kata Kunci: *Generalized fractional integrals, Muckenhoupt class, generalized weighted Morrey spaces.*

INTRODUCTION

Morrey spaces were first introduced by Charles Bradfield Morrey in 1938 (Morrey, 1938). He used the spaces to study the local behaviour of the elliptic partial diferential equation. The Morrey spaces were denoted by $\mathcal{M}^{p,\lambda} = \mathcal{M}^{p,\lambda}(\mathbb{R}^n)$ where $1 \leq p < \infty$ and $0 \leq \lambda < n$. The space $\mathcal{M}^{p,\lambda}$ was equipped with the norm

$$\|f\|_{\mathcal{M}^{p,\lambda}(\mathbb{R}^n)} = \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{r^\lambda}\left(\int_{B(a,r)} |f(y)|^p \, dy\right)^{\frac{1}{p}}.$$

The Morrey spaces can be considered as the generalization of the Lebesgue space since if we set $\lambda = 0$, we then have that $\|f\|_{\mathcal{M}^{p,\lambda}} = \|f\|_{L^p}$ and $\mathcal{M}^{p,\lambda}$ is the Lebesgue space L^p. The spaces were modified become $\mathcal{M}_q^p$ where $1 \leq p \leq q < \infty$, and the new space $\mathcal{M}_q^p$ is equpped with the norm

$$\|f\|_{\mathcal{M}_q^p} = \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{|B(a,r)|^{\frac{1}{q}-\frac{1}{p}}} \left(\int_{B(a,r)} |f(y)|^p \, dy\right)^{\frac{1}{p}}$$

as written by Gunawan *et al* (2018) and Sawano *et al* (2020). The last norm allows the other researchers to generalize the Morrey spaces. One of the generalizations is the generalized Morrey space $\mathcal{M}_\psi^p$ (Mizuhara, 1991; Nakai, 1993) where ψ is a positive function on $\mathbb{R}^n \times (0,\infty)$ as also written by Sawano *et al* (2020) and Sawano (2020). $\mathcal{M}_\psi^p$ is a set of all functions f such that $\|f\|_{\mathcal{M}_\psi^p}$ is finite where

$$\|f\|_{\mathcal{M}_q^p} = \sup_{a\in\mathbb{R}^n, r>0} \left(\frac{1}{\psi(a,r)} \int_{B(a,r)} |f(y)|^p \, dy\right)^{\frac{1}{p}}.$$

Another extension of the Morrey spaces is the weighted Morrey spaces $L^{p,\kappa}$ which were introduced by Komori & Shirai (2009) where $0 < \kappa < 1$ and w is locally integrable function taking value in $(0,\infty)$ almost everywhere. The space $L^{p,\kappa}$ is defined as a set of all functions f such that the norm

$$\|f\|_{\mathcal{M}_q^p} = \sup_{a\in\mathbb{R}^n, r>0} \left(\frac{1}{w\big(B(a,r)\big)^\kappa} \int_{B(a,r)} |f(y)|^p \, dy\right)^{\frac{1}{p}}$$

is finite where $w\big(B(a,r)\big) = \int_{B(a,r)} w(y)\, dy$. The generalization of the generalized Morrey spaces and the weighted Morrey spaces is the generalized weighted Morrey spaces as stated by Karaman *et al* (2014) and Guliyev & Mustafayev (2014). The definition of the spaces is given in the next section.

One of the important operators in harmonic analysis is the Riesz Potential or the fractional integral I_α where $0 < \alpha < n$. The operator was bounded from L^p to L^q for $1 < p < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ (Stein, 1993). Related to Morrey spaces, I_α is bounded from one Morrey space to another Morrey space under some assumptions (Adams, 1975). Moreover, I_α is bounded from one weighted Lebesgue space to another weighted Lebesgue space (Muckenhoupt and Wheeden, 1976) and bounded from one weighted Morrey space to another weighted Morrey spaces (Komori and Shirai, 2009) under some assumptions. Moreover, the operator is bounded from one generalized weighted Morrey space to generalized weighted Morrey space under some assumptions (Ramadana and Gunawan, 2022).

The generalization of fractional integrals is the generalized fractional integrals. The operators were bounded from one generalized Morrey space to another generalized Morrey space as the results by Eridani (2006). In the proof of the main results, he used the boundedness of the Hardy-Littlewood maximal operator on generalized Morrey spaces obtained by Nakai (1993). The purpose of this research is to extend the Eridani's results, that is to investigate the boundedness of the generalized fractional integrals on the generalized weighted Morrey spaces, in particular, from one generalized weighted Morrey space to anthoer generalized weighted Morrey space. The results then implies the boundedness of the generalized fractional maximal operators on the generalized weighted Morrey spaces.

DEFINITIONS AND NOTATIONS

The space we use is $\mathbb{R}^n$ equipped by the standard Euclidean distance $|\cdot|$. Let $a \in \mathbb{R}^n$ and $r > 0$. We denote $B(a,r)$ by the (open) ball centerd at a with radius r, namely

$$B(a,r) = \{x \in \mathbb{R}^n : |x-a| < r\}.$$

The integral used in this research is the Lebesgue integral. Thus, the mesure used is the Lebesgue measure. For more details about the Lebesgue measure and integration, one may refer to the book written by Royden & Fitzpatrick (2009) and Kadets (2018). For a measurable set E in $\mathbb{R}^n$, $|E|$ denotes the Lebesgue measure of the set E.

In this section, the definitions in this paper are described, namely the Muckenhoupt class, the generalized weighted Morrey spaces, the generalized fractional integrals, and generalized fractional maximal integrals. Moreover, some previous results are also stated in this section.

w is a weight on $\mathbb{R}^n$ if it is locally integrable functions on $\mathbb{R}^n$ which is taking value on $(0,\infty)$ almost everywhere. The weight class used in this research is the Muckenhoupt class A_p. The definition of A_p class is the following.

Definition 2.1. (Garcia-Cuerva & de Francia, 1985) For $1 < p < \infty$, A_p is the set of all weights w on $\mathbb{R}^n$ such that there is a constat $C > 0$ satisfying

$$\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w(y)\,dy\right)\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w(y)^{-\frac{1}{p-1}}\,dy\right)^{p-1} \le C$$

for all $a \in \mathbb{R}^n$ and $r > 0$.

The following theorem is one of the important properties of the weight class A_p.

Theorem 2.2. (Garcia-Cuerva & de Francia, 1985) *For each* $1 < p < \infty$ *and* $w \in A_p$, *there is two constants* $C_1 > 0, C_2 > 0$ *and* $\delta > 0$ *such that*

$$C_1\left(\frac{\mu(B)}{\mu(E)}\right)^{\delta} \le \frac{w(B)}{w(S)} \le C_2\left(\frac{\mu(B)}{\mu(E)}\right)^{p}$$

for all balls B *in* $\mathbb{R}^n$ *and measurable sest* $E \subseteq B$ *where*

$$w(E) = \int_E w(y)\,dy.$$

Let $1 < p < \infty$ and K be a measurable set in $\mathbb{R}^n$. The space $L^{p,w}(K)$ is set of all measurable functions f on K such that $\|f\|_{L^{p,w}(\mathrm{K})}$ is finite where

$$\|f\|_{L^{p,w}(\mathrm{K})} = \left(\int_{\mathrm{K}} |f(y)|^p w(y)dy\right)^{\frac{1}{p}}.$$

If $\mathrm{K} = \mathbb{R}^{\mathrm{n}}$, we write $L^{p,w}(\mathrm{K}) = L^{p,w}$. We see that if w is a constant almost everywhere on $\mathbb{R}^n$, then $L^{p,w} = L^p$. Next, we define the generalized weighted Morrey spaces.

Definition 2.3. (Akbulut, 2014) Let $1 \le p < \infty$, w be a weight on $\mathbb{R}^n$, and ψ be a positive function on $\mathbb{R}^n \times (0,\infty)$. The generalized weighted Morrey spaces $\mathcal{M}_{\psi}^{p,w}$ is a set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{\mathcal{M}_{\psi}^{p,w}} = \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi(a,r)} \frac{1}{w\big(B(a,r)\big)^{\frac{1}{p}}} \|f\|_{L^{p,w}(B(a,r))} < \infty.$$

Note that If w is a constant a.e., and $\psi(a,r) = r^{-\frac{n}{q}}$ where $1 \le p \le p < \infty$, then $\mathcal{M}_{\psi}^{p,w}$ is the classical Morrey space. If $\psi(a,r) = w\big(B(a,r)\big)^{\frac{\kappa-1}{p}}$ where $0 < \kappa < 1$, then $\mathcal{M}_{\psi}^{p,w}$ is the weighted

Morrey spaces $L^{p,\kappa}(w)$. In this research, our main results are related to the boundedness properties of certain operators on the generalized weighted Morreys spaces. We assume that ψ is a function of r. The definition is given as follows.

Definition 2.4. Let $1 < p < \infty$, w be a weight on $\mathbb{R}^n$, and ψ be a postive functions on $(0, \infty)$. The generalized weighted Morrey spaces $\mathcal{M}_{\psi}^{p,w}$ is a set of all measurable function f on $\mathbb{R}^n$ such that

$$\|f\|_{\mathcal{M}_{\psi}^{p,w}} = \sup_{a \in \mathbb{R}^n, r>0} \frac{1}{\psi(r)} \frac{1}{w\big(B(a,r)\big)^{\frac{1}{p}}} \|f\|_{L^{p,w}(B(a,r))} < \infty.$$

The Definition 2.3. is more general than the Definition 2.4. Next, we consider some operators used in this article.

Definition 2.5. (Grafakos, 2014; Sawano et al, 2020) The Hardy-Littlewood maximal operator M is defined as

$$M(f)(x) = \sup_{r>0} \frac{1}{|B(a,r)|} \int_{B(a,r)} |f(y)|\, dy, \qquad x \in \mathbb{R}^n$$

for all suitable functions f on $\mathbb{R}^n$.

Related to the generalized weighted Morrey space, we have the theorem 2.6 as follows.

Theorem 2.6. (Ramadana and Gunawan, 2022) *Let $1 < p < \infty$ and $w \in A_p$. Suppose that ψ_1 and ψ_2 are two positive functions on $\mathbb{R}^n \times (0, \infty)$ for which there is a constant $C > 0$ such that*

$$\int_r^\infty \psi_1(a,t) \frac{dt}{t} \le C\, \psi_2(a,r)$$

for all $a \in \mathbb{R}^n$ and $r > 0$. Then, M is bounded from $\mathcal{M}_{\psi_1}^{p,w}$ to $\mathcal{M}_{\psi_2}^{p,w}$.

Definition 2.7. (Sawano et al, 2009) Let ρ be nonnegative function taking value on $[0, \infty)$. The generalized fractional integral I_ρ is defined as

$$I_\rho(f)(x) = \int_{\mathbb{R}^n} \frac{\rho(|x-y|)}{|x-y|^n} f(y) dy, \qquad x \in \mathbb{R}^n$$

for all suitable functions f on X.

If $\rho(t) = t^\alpha, t \ge 0$ where $0 < \alpha < n$, we have that I_ρ is the fractional integral I_α as investigated by Ramadana & Gunawan (2022).

Definition 2.8. (Kucukaslan, 2020) For nonnegative function ρ taking value on $[0, \infty)$, the generalized fractional maximal operator M_ρ is defined as

$$M_\rho(f)(x) = \sup_{r>0} \frac{\rho(r)}{r^n} \int_{B(x,r)} |f(y)|\, dy, \qquad x \in \mathbb{R}^n$$

for all suitable functions f on $\mathbb{R}^n$.

The relation betweeen I_ρ and M_ρ is the inequality

$$M_\rho(f)(x) \le C\, I_\rho(|f|)(x). \tag{1}$$

By the inequality (1), the boundedness of I_ρ on the generalized weighted Morrey spaces implies the boundedness of M_ρ on the generalized weighted Morrey spaces under the same assumptions.

MAIN RESULT

In this section, we state the main result and its corollaries. The following are the main results in this research.

Theorem 3.1. *Let ρ be a nonnegative function on $[0,\infty)$ and $w \in A_p$ where $1 < p < \infty$. Suppose that ψ_1 and ψ_2 are two positive functions on $(0,\infty)$, and the following conditions hold:*

(i) *for* $\frac{1}{2} \le \frac{t}{r} \le 2$,

$$\frac{1}{C_1} \le \frac{\psi_1(t)}{\psi_1(r)} \le C_1,$$

and

$$\frac{1}{C_2} \le \frac{\rho(t)}{\rho(r)} \le C_2,$$

(ii) *for all* $r > 0$,

$$\int_r^\infty \psi_1(r)\frac{dt}{t} \le C_3\psi_1(r),$$

and

$$\psi_1(r)\int_0^r \frac{\rho(t)}{t}dt + \int_r^\infty \rho(t)\psi_1(t)\frac{dt}{t} \le C_4\psi_2(r)$$

where A_1, A_2, A_3, and A_4 are positive constants independent of the choice of $a \in \mathbb{R}^n$ and $r > 0$. Then, I_ρ is bounded from $\mathcal{M}_{\psi_1}^{p,w}$ to $\mathcal{M}_{\psi_2}^{p,w}$.

By inequality (1), we have the following corollary about the boundedness of the generalized fractional maximal operators on the generalized weighted Morrey spaces.

Corollary 3.2. *Let ρ be a nonnegative function on $[0,\infty)$ and $w \in A_p$ where $1 < p < \infty$. Suppose that ψ_1 and ψ_2 are two positive functions on $(0,\infty)$, and the following conditions hold:*

(i) *for* $\frac{1}{2} \le \frac{t}{r} \le 2$,

$$\frac{1}{C_1} \le \frac{\psi_1(t)}{\psi_1(r)} \le C_1,$$

and

$$\frac{1}{C_2} \le \frac{\rho(t)}{\rho(r)} \le C_2,$$

(ii) *for all $a \in \mathbb{R}^n$ and $r > 0$,*

$$\int_r^\infty \psi_1(t)\frac{dt}{t} \le C_3\psi_1(r),$$

and

$$\psi_1(r)\int_0^r \frac{\rho(t)}{t}dt + \int_r^\infty \rho(t)\psi_1(t)\frac{dt}{t} \le C_4\psi_2(r)$$

where A_1, A_2, A_3, and A_4 are positive constants independent of the choice of $a \in \mathbb{R}^n$ and $r > 0$. Then, M_ρ is bounded from $\mathcal{M}_{\psi_1}^{p,w}$ to $\mathcal{M}_{\psi_2}^{p,w}$.

By using Theorem 3.1 and combining with Corollary 3.2 as well as by taking $\psi_1 = \psi_2$, we have Corollary 3.3 as follows.

Corollary 3.3. *Let ρ be a nonnegative function on $[0,\infty)$ and $w \in A_p$ where $1 < p < \infty$. Suppose that ψ is positive function on $(0,\infty)$, and the following conditions hold:*

(i) *for* $\frac{1}{2} \le \frac{t}{r} \le 2$,

$$\frac{1}{C_1} \le \frac{\psi(t)}{\psi(r)} \le C_1,$$

and

$$\frac{1}{C_2} \le \frac{\rho(t)}{\rho(r)} \le C_2,$$

(ii) *for all $a \in \mathbb{R}^n$ and $r > 0$,*

$$\int_0^r \frac{\rho(t)}{t} dt + \frac{1}{\psi(r)} \int_r^\infty \big(1 + \rho(t)\big)\psi(t) \frac{dt}{t} \le C_3$$

where A_1, A_2, and A_3 are positive constants independent of the choice of $a \in \mathbb{R}^n$ and $r > 0$. Then, I_ρ and M_ρ is bounded on $\mathcal{M}_\psi^{p,w}$.

The following corollary is about the boundedness of the generalized fractional integrals on the generalized Morrey spaces. It is based on the fact that the function $w \equiv 1$ a.e. is a weight in A_p for $1 < p < \infty$.

Corollary 3.4. (Eridani, 2006) *Let ρ be a nonnegative function on $[0,\infty)$ and $1 < p < \infty$. Suppose that ψ_1 and ψ_2 are two positive functions on $(0,\infty)$, and the following conditions hold:*

(i) *for* $\frac{1}{2} \le \frac{t}{r} \le 2$,

$$\frac{1}{C_1} \le \frac{\psi_1(t)}{\psi_1(r)} \le C_1,$$

and

$$\frac{1}{C_2} \le \frac{\rho(t)}{\rho(r)} \le C_2,$$

(ii) *for all $r > 0$,*

$$\int_r^\infty \psi_1(r) \frac{dt}{t} \le C_3 \psi_1(r),$$

and

$$\psi_1(r) \int_0^r \frac{\rho(t)}{t} dt + \int_r^\infty \rho(t)\psi_1(t) \frac{dt}{t} \le C_4 \psi_2(r)$$

where A_1, A_2, A_3, and A_4 are positive constants independent of the choice of $a \in \mathbb{R}^n$ and $r > 0$. Then, I_ρ and M_ρ is bounded from $\mathcal{M}_{\psi_1}^{p,w}$ to $\mathcal{M}_{\psi_2}^{p,w}$.

PROOFS OF THE MAIN RESULTS

Let $1 < p < \infty$. p' is the conjugate of p, which is the unique number in $(1,\infty)$ satisfying $1/p + 1/\mathrm{p}' = 1$. From now on, C denote a positive constant which may vary from line to line. In general, $C > 0$ depends on n and p. Before proving the Theorem 3.1, we first prove the following theorem about the estimates for the generalized fractional integrals.

Theorem 4.1. *Let ρ be a nonnegative function on $[0,\infty)$ and $w \in A_p$ where $1 < p < \infty$. Suppose that ψ_1 and ψ_2 are two positive functions on $(0,\infty)$, and the following conditions hold:*

(i) *for* $\frac{1}{2} \leq \frac{t}{r} \leq 2$,

$$\frac{1}{C_1} \leq \frac{\psi_1(t)}{\psi_1(r)} \leq C_1,$$

and

$$\frac{1}{C_2} \leq \frac{\rho(t)}{\rho(r)} \leq C_2,$$

(ii) *for all* $r > 0$,

$$\int_r^\infty \psi_1(r) \frac{dt}{t} \leq C_3 \psi_1(r),$$

where A_1, A_2, A_3, *and* A_4 *are positive constants independent of the choice of* $a \in \mathbb{R}^n$ *and* $r > 0$. *Then, there is a constant* $C > 0$ *such that*

$$\left|I_\rho(f)(x)\right| \leq C \left[M(f)(x) \int_0^r \rho(t) \frac{dt}{t} + \|f\|_{\mathcal{M}_{\psi_1}^{p,w}} \int_r^\infty \rho(t) \psi_1(t) \frac{dt}{t}\right]$$

for all $x \in \mathbb{R}^n, r > 0$, *and* $\|f\|_{\mathcal{M}_{\psi_1}^{p,w}}$.

Proof Theorem 4.1: Let $r > 0$. Then, for all $x \in \mathbb{R}^n$ we have

$$\begin{aligned} I_\rho(f)(x) &= \int_{\mathbb{R}^n} \frac{\rho(|x-y|)}{|x-y|^n} f(y)\, dy \\ &= \int_{\mathbb{R}^n} \frac{\rho(|x-y|)}{|x-y|^n} f_1(y)\, dy + \int_{\mathbb{R}^n} \frac{\rho(|x-y|)}{|x-y|^n} f_2(y)\, dy \\ &= \int_{|x-y|<r} \frac{\rho(|x-y|)}{|x-y|^n} f(y)\, dy + \int_{|x-y|\geq r} \frac{\rho(|x-y|)}{|x-y|^n} f(y)\, dy. \end{aligned}$$

Note that by the assumption (i) and integral subtitution, we have that

$$\int_{2^k r}^{2^{k+1} r} \frac{\rho(t)}{t} dt = \int_1^2 \frac{\rho\left(2^k r\, t\right)}{2^k r\, t} 2^k r\, dt = \int_{\frac{1}{2}}^1 \rho\left(2^k r\, t\right) \frac{dt}{t} \geq C\, \rho\left(2^k r\right).$$

By the same way, we have

$$\int_{2^k r}^{2^{k+1} r} \frac{\rho(t)\psi_1(t)}{t} dt \geq C\, \rho\left(2^k r\right) \psi_1\left(2^k r\right).$$

We let

$$\mathcal{I}_1^r(x) = \int_{|x-y|<r} \frac{\rho(|x-y|)}{|x-y|^n} f(y)\, dy$$

and

$$\mathcal{I}_2^r(x) = \int_{|x-y|\geq r} \frac{\rho(|x-y|)}{|x-y|^n} f(y)\, dy.$$

Thus,

$$I_\rho(f)(x) = \mathcal{I}_1^r(x) + \mathcal{I}_2^r(x).$$

In general, we write $I_\rho(f) = \mathcal{I}_1^r + \mathcal{I}_2^r$. We shall find the estimates for $\mathcal{I}_1^r$ and $\mathcal{I}_2^r$. First, we estimate $\mathcal{I}_1^r$. By the previous estimates and assumption (i) related to ρ, we have

$$|\mathcal{I}_1^r(x)| \leq \int_{|x-y|<r} \frac{\rho(|x-y|)}{|x-y|^n} f(y)\, dy$$

$$
\begin{aligned}
&= \sum_{k=-\infty}^{-1} \int_{2^k r \le |x-y| < 2^{k+1} r} \frac{\rho(|x-y|)}{|x-y|^n} f(y)\, dy \\
&\le C \sum_{k=-\infty}^{-1} \int_{2^k r \le |x-y| < 2^{k+1} r} \frac{\rho(2^k r)}{|x-y|^n} f(y)\, dy \\
&\le C \sum_{k=-\infty}^{-1} \rho(2^k r) \int_{|x-y| < 2^{k+1} r} \frac{f(y)}{|x-y|^n}\, dy \\
&\le C \sum_{k=-\infty}^{-1} \frac{\rho(2^k r)}{(2^k r)^n} \int_{|x-y| < 2^{k+1} r} f(y)\, dy \\
&= C \sum_{k=-\infty}^{-1} \frac{\rho(2^k r)}{|B(x, 2^{k+1} r)|} \int_{|x-y| < 2^{k+1} r} f(y)\, dy \\
&\le C \sum_{k=-\infty}^{-1} \rho(2^k r)\, Mf(x) \\
&\le C\, Mf(x) \sum_{k=-\infty}^{-1} \int_{2^k r}^{2^{k+1} r} \frac{\rho(t)}{t}\, dt \\
&= C\, Mf(x) \int_0^r \frac{\rho(t)}{t}\, dt
\end{aligned}
$$

Next, we estimate $\mathcal{I}_2(x)$. By the assumption that $w \in A_p$, then Hölder's inequality implies the following inequalities.

$$
\begin{aligned}
&\mathcal{I}_2(x) \\
&\le \int_{|x-y| \ge r} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)|\, dy \\
&\le \sum_{k=0}^{\infty} \int_{2^k r \le |x-y| < 2^{k+1} r} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)|\, dy \\
&\le C \sum_{k=0}^{\infty} \int_{2^k r \le |x-y| < 2^{k+1} r} \frac{\rho(2^{k+1} r)}{|x-y|^n} |f(y)|\, dy \\
&\le C \sum_{k=0}^{\infty} \frac{\rho(2^{k+1} r)}{(2^{k+1} r)^n} \int_{|x-y| < 2^{k+1} r} |f(y)|\, dy \\
&= C \sum_{k=0}^{\infty} \rho(2^{k+1} r) \frac{1}{|B(a, 2^{k+1})|} \int_{|x-y| < 2^{k+1} r} |f(y)|\, dy \\
&= C \sum_{k=0}^{\infty} \rho(2^{k+1} r) \frac{1}{|B(a, 2^{k+1})|} \left(\int_{|x-y| < 2^{k+1} r} |f(y)|^p w(y)\, dy \right)^{\frac{1}{p}} \left(\int_{|x-y| < 2^{k+1} r} w(y)^{-p'}\, dy \right)^{\frac{1}{p'}} \\
&= C \sum_{k=0}^{\infty} \rho(2^{k+1} r) \frac{1}{w\left(B(a, 2^{k+1} r)\right)^{\frac{1}{p}}} \|f\|_{L^{p,w}\left(B(x, 2^{k+1} r)\right)} \frac{1}{|B(a, 2^{k+1})|} \left(\int_{|x-y| < 2^{k+1} r} w(y)\, dy \right)^{\frac{1}{p}} \\
&\cdot \left(\int_{|x-y| < 2^{k+1} r} w(y)^{-p'}\, dy \right)^{\frac{1}{p'}} \\
&\le C \sum_{k=0}^{\infty} \rho(2^{k+1} r) \psi_1(a, 2^{k+1} r)\, \|f\|_{\mathcal{M}_{\psi_1}^{p,w}}
\end{aligned}
$$

$$\leq C\,\|f\|_{\mathcal{M}_{\psi_1}^{p,w}} \sum_{k=0}^{\infty} \int_{2^{k+1}r}^{2^{k+2}r} \rho(t)\psi_1(x,t)\frac{dt}{t}$$

$$\leq C\,\|f\|_{\mathcal{M}_{\psi_1}^{p,w}} \int_{r}^{\infty} \rho(t)\psi_1(x,t)\frac{dt}{t}$$

By combining with the first estimates, we have that

$$|I_\rho(f)(x)| \leq C\left[M(f)(x)\int_0^r \rho(t)\frac{dt}{t} + \|f\|_{\mathcal{M}_{\psi_1}^{p,w}}\int_r^\infty \rho(t)\psi_1(t)\frac{dt}{t}\right]$$

as desired. ∎

Now, we are ready to prove the Theorem 3.1.

Proof of Theorem 3.1: By Theorem 4.1, we have

$$|I_\rho(f)(x)| \leq C\left[M(f)(x)\int_0^r \rho(t)\frac{dt}{t} + \|f\|_{\mathcal{M}_{\psi_1}^{p,w}}\int_r^\infty \rho(t)\psi_1(t)\frac{dt}{t}\right].$$

For every ball $B(a,r)$, we take the norm $\|\cdot\|_{L^{p,w}(B(a,r))}$ on the both sides in the last inequality, then by Minkowski's inequality and the assumptions in the theorem, we obtain

$$\begin{aligned}
&\|I_\rho(f)\|_{L^{p,w}(B(a,r))}\\
&\leq C\left[\|M(f)\|_{L^{p,w}(B(a,r))}\int_0^r \rho(t)\frac{dt}{t} + w(B(a,r))^{\frac{1}{p}}\|f\|_{\mathcal{M}_{\psi_1}^{p,w}}\int_r^\infty \rho(t)\psi_1(t)\frac{dt}{t}\right]\\
&\leq C\left[\|M(f)\|_{L^{p,w}(B(a,r))}\frac{\psi_2(r)}{\psi_1(r)} + w(B(a,r))^{\frac{1}{p}}\,\psi_2(r)\,\|f\|_{\mathcal{M}_{\psi_1}^{p,w}}\right].
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{1}{\psi_2(r)}\frac{1}{w(B(a,r))^{\frac{1}{p}}}\,\|I_\rho(f)\|_{L^{p,w}(B(a,r))}\\
&\qquad\leq C\left[\frac{1}{\psi_1(r)}\frac{1}{w(B(a,r))^{\frac{1}{p}}}\,\|M(f)\|_{L^{p,w}(B(a,r))} + \|f\|_{\mathcal{M}_{\psi_1}^{p,w}}\right]
\end{aligned}$$

Taking the supremum over $a \in \mathbb{R}^n$ and $r > 0$, we get

$$\|I_\rho(f)\|_{\mathcal{M}_{\psi_1}^{p,w}} \leq C\left[\|M(f)\|_{\mathcal{M}_{\psi_1}^{p,w}} + \|f\|_{\mathcal{M}_{\psi_1}^{p,w}}\right]$$

By theorem 2.4,

$$\|M(f)\|_{\mathcal{M}_{\psi_1}^{p,w}} \leq C\,\|f\|_{\mathcal{M}_{\psi_1}^{p,w}}, \qquad f \in \mathcal{M}_{\psi_1}^{p,w}.$$

Therefore,

$$\|I_\rho(f)\|_{\mathcal{M}_{\psi_1}^{p,w}} \leq C\,\|f\|_{\mathcal{M}_{\psi_1}^{p,w}}$$

for all $f \in \mathcal{M}_{\psi_1}^{p,w}$ where $C > 0$ is independent of f. This completes the proof of Theorem 3.1. ∎

CONCLUSIONS

The main result shows that the generalized fractional integrals I_ρ are bounded from the generalized weighted Morrey space $\mathcal{M}_{\psi_1}^{p,w}$ to another Morrey space $\mathcal{M}_{\psi_2}^{p,w}$, e.i., $\|I_\rho(f)\|_{\mathcal{M}_{\psi_2}^{p,w}} \leq C\,\|f\|_{\mathcal{M}_{\psi_1}^{p,w}}$ for all $f \in \mathcal{M}_{\psi_1}^{p,w}$ under some assumptions. The boundedness property then implies the same result for the generalized fractional maximal operator M_ρ, namely, M_ρ is bounded from one generalized weighted Morrey space $\mathcal{M}_{\psi_1}^{p,w}$ to another generalized weighted Morrey space $\mathcal{M}_{\psi_2}^{p,w}$ under the same assumptions. For a future research, it is interesting to investigate the boundedness of I_ρ on more general spaces, such as the generalized weighted Orlicz-Morrey space.

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