Daya Matematis : Jurnal Inovasi Pendidikan Matematika
Volume 10 Nomor 3 December 2022 Hal. 223-230
p-ISSN:2541-4232 dan e-ISSN: 2354-7146

THE ORTOGONAL PROPERTY OF DIRECTIONAL SHORT-TIME QUATERNION FOURIER TRANSFORM

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Abstract
This study will examine the orthogonal properties of Directional Short-time quaternion Fourier transform (DSTQFT) which is a further study of DSTFT with an expansion in the form of a function that has a Quaternion value. The orthogonal properties of DSTQFT are obtained by combining the orthogonal properties of DSTFT and the Fourier quaternion transform (QFT). Based on the results of the study, it was found that the orthogonal nature of the Directional Short-Time quaternion Fourier Transform (DSTQFT) is different from the orthogonal nature of the Directional Short-Time Fourier Transform (DSTFT). In addition, there are three important consequences of orthogonality in TFQSTB as a result of the quaternion-valued function.

Keywords: Orthogonal properties, Fourier quaternion transform (QFT), Directional Short-Time Fourier Transform (DSTFT), Directional Short-Time quaternion Fourier Transform (DSTQFT).

INTRODUCTION

As the theory and applications of the Fourier transform (TF) develop in the field of signal analysis, it turns out that analysts in this field think that TF has not been able to meet the needs of signal analysis. The initial solution to this problem is the Fourier Short-Time Transform (TFST). Furthermore, in 2013, Giv developed the TFST theory into a Directed Fourier Short-Time Transform (TFSTB). (J. Math. Anal. Appl. 399:100-107, 2013). Giv examines the basic properties of TFSTB, especially the orthogonal properties as outlined in the theorem.
There are various types of numbers such as real numbers, complex numbers and others. Complex numbers are numbers that can be written in the form a+ib as ordered pairs in the Cartesian plane (a,b). Because complex numbers are only used in planes, the idea arose to extend complex numbers to 3-dimensional space, which is used for calculations that involve 3-dimensional rotation, such as 3-dimensional computer graphics and crystallographic analysis of numbers. These numbers are called quaternions. Quaternion was first introduced by W.R. Hamilton namely the expansion of complex numbers (Dospra, 2015). It differs from a complex number in that it involves three imaginary units which are not commutative. Often times, quaternion numbers are also used in functions that have quaternion values. From this breakthrough, a Fourier quaternion transform appears as an extension of the Fourier transform. Further research has been conducted on the Short-Time Directional Fourier Transform as a solution to signal requirements with time resolution.

In this research, the TFSTB will be studied further with an extension to the Quaternion domain and this new transformation is called the Short-Time Fourier Quaternion Transform (TFQSTB). By combining the orthogonal properties of the TFSTB and the Fourier Quaternion Transformation (TFQ), we obtain the Orthogonal properties of the Short-Time Fourier Quaternion Transformation (TFQSTB).

This research was conducted by first collecting references related to the TFSTB and Quaternion, secondly conducting a literature review of the references obtained, and thirdly formulating the orthogonal nature of the TFQSTB.

1.1. Fourier Transform

In this section, will be introduce the definition of Fourier transform and the basic properties.

Definition 1. Let $f \in L^2(\mathbb{R})$, the Fourier transform of complex function f is defined by

$$\mathcal{F}\{f\}(\omega) = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx, \qquad (1.1)$$

with $e^{-i\omega t} = cos\omega t - i\ sin\omega t$, so equation (1.1) can be written by

$$\mathcal{F}\{f\}(\omega) = \int_{-\infty}^{\infty} f(x)\cos \omega x\, dx - i\int_{-\infty}^{\infty} f(x)\sin \omega x\, dx.$$

The basic properties of the Fourier transform are as follows:

a. Linearity Property
   Let $f \in L^1(\mathbb{R})$ and for all $\omega \in \mathbb{R}$, then
   $$\mathcal{F}\{\alpha f + \beta g\}(\omega) = \alpha\mathcal{F}\{f\}(\omega) + \beta\mathcal{F}\{g\}(\omega).$$
   Where α and β is constant real numbers.

b. Multiplication Property
   Let $f \in L^1(\mathbb{R})$ and for all constant $k \in \mathbb{C}$, then
   $$\mathcal{F}\{kf\}(\omega) = k\mathcal{F}\{f\}(\omega).$$

c. Translasi Property
   Let $f \in L^1(\mathbb{R})$ and translation $\tau_k f(t) = f(t-k)$, then
   $$\mathcal{F}\{\tau_k f\}(\omega) = e^{-i\omega k}\mathcal{F}\{f\}(\omega).$$

d. Modulation Property
   Let $f \in L^1(\mathbb{R})$ and $\omega_0 \in \mathbb{R}$. If $M_{\omega_0} f(t) = e^{i\omega_0 t} f(t)$, then
   $$\mathcal{F}\{M_{\omega_0} f\}(\omega) = \mathcal{F}\{f\}(\omega - \omega_0).$$

e. The Property of Translation and Modulation
   Let $f \in L^1(\mathbb{R})$ and $k, \omega_0 \in \mathbb{R}$. If $M_{\omega_0}\tau_k f(t) = e^{i\omega_0 t} f(t-k)$, then
   $$\mathcal{F}\{M_{\omega_0}\tau_k f(t)\}(\omega) = e^{-i(\omega-\omega_0)k}\mathcal{F}\{f\}(\omega - \omega_0).$$

1.2. Quaternion algebra

The quaternion algebra was first invented by Sir W. R. Hamilton in 1843 and is denoted by $\mathbb{H}$ in his honor. It is an extension of complex numbers to a 4D algebra. Every element of $\mathbb{H}$ is a linear combination of a real scalar and three orthogonal imaginary units (denoted by i, j and k) with real coefficients

$$\mathbb{H} = \{q \mid q = a_0 + a_1 i + a_2 j + a_3 k\} \text{ with } a_0, a_1, a_2, a_3 \in \mathbb{R}$$

In this case the elements $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ obey Hamilton's multiplication rules

$i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$.

Because $\mathbb{H}$ non-commutative, one cannot directly extend various results on complex numbers to quaternions. For simplicity, we express a quaternion q as sum of a scalar a_0 and a pure 3D quaternion $\mathbf{q}$

$$\mathrm{q} = a_0 + \boldsymbol{q} = a_0 + a_1 i + a_2 j + a_3 k.$$

Where the scalar part is also denoted Sc(q) = a_0. The conjugate of a quaternion q is obtained by changing the sign of the pure part, i.e.

$$\bar{q} = (a_0, -\mathbf{q}) = a_0 - a_1 i - a_2 j - a_3 k.$$

The quaternion conjugation is a linear anti-involution

$$\bar{\bar{p}} = p, \qquad \overline{p+q} = \bar{p} + \bar{q}, \qquad \overline{pq} = \bar{q}\bar{p}, \qquad \forall p, q \in \mathbb{H}$$

Given a quaternion q and its conjugate, we can easily check that the following properties are correct

$$a_0 = \frac{1}{2}(q + \bar{q}), \quad \boldsymbol{q} = \frac{1}{2}(q - \bar{q}), \quad q = -\bar{q} \leftrightarrow q = \boldsymbol{q}.$$

The multiplication of the two quaternions $q = a_0 + \boldsymbol{q}$ and $p = b_0 + \boldsymbol{p}$ can be expressed as

$$\mathrm{qp} = \mathrm{a}_0 b_0 + \boldsymbol{q} \cdot \boldsymbol{p} + a_0 \boldsymbol{p} + b_0 \boldsymbol{q} + \boldsymbol{q} \times \boldsymbol{p}.$$

Where we recognize the scalar product $\boldsymbol{q} \cdot \boldsymbol{p} = -(a_1 b_1 + a_2 b_2 + a_3 b_3)$ and the anti-symmetric cross type product

$$\boldsymbol{q} \times \boldsymbol{p} = \boldsymbol{i}(a_2 b_3 - a_3 b_2) + \boldsymbol{j}(a_3 b_1 - a_1 b_3) + \boldsymbol{k}(a_1 b_2 - a_2 b_1).$$

The scalar part of the product is Scal$(qp) = \mathrm{a}_0 b_0 + \boldsymbol{q} \cdot \boldsymbol{p}$, and the pure part is

$$a_0 \boldsymbol{p} + b_0 \boldsymbol{q} + \boldsymbol{q} \times \boldsymbol{p}.$$

The multiplication of a quaternion q and its conjugate can be expressed as

$$\bar{q}\mathrm{q} = \mathrm{q}\bar{q} = a_0 a_0 - \boldsymbol{q} \cdot \boldsymbol{q} + a_0(-\boldsymbol{q}) + a_0 \boldsymbol{q} + \boldsymbol{q} \times (-\boldsymbol{q})$$

$$= a_0{}^2 + a_1{}^2 + a_2{}^2 + a_3{}^2.$$

This will lead to the modulus $|\mathrm{q}|$ of a quaternion q defined as

$$|\mathrm{q}| = \sqrt{\mathrm{q}\bar{q}} = \sqrt{a_0{}^2 + a_1{}^2 + a_2{}^2 + a_3{}^2}$$

Using the conjugate of q and the modulus of a quaternion q, we can define the inverse of $q \in \mathbb{H}\backslash\{0\}$ as

$$\mathrm{q}^{-1} = \frac{\bar{q}}{|\mathrm{q}|^2}.$$

1.3 Quaternion Fourier Transform (QFT)

It is natural to extend the Fourier transform to quaternion algebra. These extensions are broadly called the quaternionic Fourier transform (QFT). Due to the non-commutative properties of quaternions, there are three different types of QFT: a left-sided QFT, a right-sided QFT and a two-sided QFT. Below we introduce the definition of two-sided QFT.

Definition 2. (Bahri, 2008) The quaternion Fourier Transform (QFT) for all $f \in L^1(\mathbb{R}^2;\mathbb{H})$ $\mathcal{F}_q\{f\}: \mathbb{R}^2 \to \mathbb{H}$ is defined by

$$\mathcal{F}_q\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} f(x) e^{-i\omega_1 \cdot x_1} e^{-j\omega_2 \cdot x_2}\, d^2\boldsymbol{x} \tag{1.2}$$

with $\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2,\ \boldsymbol{\omega} = \omega_1\boldsymbol{e}_1 + \omega_2\boldsymbol{e}_2$ and the result of exponential multiplication quaternion $e^{-i\omega_1 \cdot x_1} e^{-j\omega_2 \cdot x_2}$ is a kernel Fourier quaternion.

1.4 Quaternion Windowed Fourier Transform (QWTF)

In this section, we introduce the definition of the quaternion windowed Fourier transform (QWTF) and provide its orthogonal property.

Definition 3. Let $\phi \in S(\mathbb{R}^2;\mathbb{H})\backslash\{0\}$ be a quaternion window function. The QWFT of $h \in S(\mathbb{R}^2;\mathbb{H})$ with respect to ϕ, is given by

$$G_\phi h(\boldsymbol{u},\boldsymbol{b}) = \int_{\mathbb{R}^2} h(x)\overline{\phi(\boldsymbol{x}-\boldsymbol{b})}\, \mathrm{e}^{-\mathrm{i}u_1x_1}\mathrm{e}^{-\mathrm{j}u_2x_2}\, \mathbf{dx}. \tag{1.3}$$

Theorem 1. (orthogonal property) Let $\phi, \psi \in S(\mathbb{R}^2;\mathbb{H})\backslash\{0\}$ be two windowed quaternion function. For $h, g \in S(\mathbb{R}^2;\mathbb{H})$, we have

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G_\phi h(\boldsymbol{u},\boldsymbol{b})\,\overline{G_\psi h(\boldsymbol{u},\boldsymbol{b})}\, \mathbf{db}\, \mathbf{du} = \left(h(\overline{\phi},\overline{\psi}), g\right). \tag{1.4}$$

for which

$$(f,g) = \int_{\mathbb{R}^2} f(\boldsymbol{x})\, \overline{g(\boldsymbol{x})}\, d\boldsymbol{x}, \qquad d\boldsymbol{x} = dx_1\, dx_2$$

Furthermore, we obtain the following important consequences

i) If $\phi = \psi$, then

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G_\phi h(\boldsymbol{u},\boldsymbol{b})\,\overline{G_\phi h(\boldsymbol{u},\boldsymbol{b})}\, \mathbf{db}\, \mathbf{du} = \|\phi\|_2^2 (h,g)$$

ii) If $h = g$, then

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G_{\phi}h(\boldsymbol{u},\boldsymbol{b})\,\overline{G_{\psi}h(\boldsymbol{u},\boldsymbol{b})}\,\mathrm{d}\mathbf{b}\,\mathrm{d}\mathbf{u} = (h(\bar{\phi},\bar{\psi}),h)$$

iii) If $h = g$ dan $\phi = \psi$, then

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G_{\phi}h(\boldsymbol{u},\boldsymbol{b})\,\overline{G_{\phi}h(\boldsymbol{u},\boldsymbol{b})}\,\mathrm{d}\mathbf{b}\,\mathrm{d}\mathbf{u} = \|\phi\|_2^2\|h\|_2^2$$

1.3 Directional short-time Fourier transform (DSTFT)

In this part, we introduce the formal definition of the Directional short-time Fourier transform (DSTFT) and orthogonal properties.

Definition 3. *(Giv, 2013.) Let $g \in L^{\infty}(\mathbb{R}^n)$ be a non-zero function. For every $f \in L^1(\mathbb{R}^n)$, we denote the directional short-time Fourier transform (DSTFT) of f with respect to g denoted by $\mathfrak{D}_g f$ and defined it as a function on $S_{n-1} \times \mathbb{R}^n \times \mathbb{R}^n$ via*

$$\mathfrak{D}_g f\,(\xi,x,\omega) = \int_{\mathbb{R}^n} f(t)\,\overline{g(\xi\cdot t - x)}\,e^{2\pi i t\cdot\omega}dt. \tag{1.5}$$

Here we assume that $g \in L^{\infty}(\mathbb{R}^n)$, for every $(\xi,x) \in S^{n-1}\times\mathbb{R}$.

Theorem 2. *(orthogonal property) Suppose* $g_1, g_2 \in L^{\infty}(\mathbb{R})$ and $f_1, f_2 \in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ If at least one of the $g_i' s$ is in $L^1(\mathbb{R})$ for $\Delta \in (\xi, x, \omega)$, *then the following orthogonality relation holds.*

$$\int_{\Delta} D_{g_1}f_1(\xi,x,\boldsymbol{\omega})\overline{D_{g_2}f_2(\xi,x,\boldsymbol{\omega})}\,dxd\boldsymbol{\omega} = \langle f_1,f_2\rangle\langle g_2,g_1\rangle. \tag{1.6}$$

In particular, if $g \in L^1(\mathbb{R})\cap L^{\infty}(\mathbb{R})$ and $f_1, f_2 \in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$, then $\mathfrak{D}_g f \in L^2(\Delta)$ and

$$\left\|\mathfrak{D}_g f\right\|_2 \le \|g\|_2\|f\|_2.$$

RESULT AND DISCUSSION

Following are the main results of this paper, namely the orthogonal properties of the directional short-time quaternion Fourier transform (DSTQFT) as shown by the following theorem.

Definition 4. *Suppose $g \in \mathrm{L}^{\infty}(\mathbb{R})$ and for all $\mathrm{f} \in \mathrm{L}^2(\mathbb{R}^2;\mathbb{H})$. DSTQFT with respect to g, is define by*

$$\mathrm{D}_{\mathrm{r}_g} f(\xi,x,\boldsymbol{\omega}) = \int_{\mathbb{R}^2} f(\mathbf{t})\overline{g(\xi\cdot\mathbf{t} - x)}\,\mathrm{e}^{-2\pi i t_1\omega_1}\mathrm{e}^{-2\pi j t_2\omega_2}\,\mathrm{dt_1dt_2}, \tag{1.7}$$

where $(\xi,\mathrm{x},\boldsymbol{\omega}) \in \mathrm{S}\times\mathbb{R}\times\mathbb{R}^2$, $\mathrm{S}\subseteq\mathbb{R}^2$ and $|\mathrm{S}| = 1$

with

$$f(\boldsymbol{t}) = f_0(\boldsymbol{t}) + if_1(\boldsymbol{t}) + jf_2(\boldsymbol{t}) + kf_3(\boldsymbol{t})$$

$$g(\boldsymbol{t}) = g_0(\boldsymbol{t}) + ig_1(\boldsymbol{t}) + jg_2(\boldsymbol{t}) + kg_3(\boldsymbol{t})$$

$$g(\xi \cdot \boldsymbol{t} - x) = g_0(\xi \cdot \boldsymbol{t} - x) + ig_1(\xi \cdot \boldsymbol{t} - x) + jg_2(\xi \cdot \boldsymbol{t} - x) + kg_3(\xi \cdot \boldsymbol{t} - x)$$

$$\overline{g(\xi \cdot \boldsymbol{t} - x)} = g_0(\xi.\boldsymbol{t} - x) - ig_1(\xi \cdot \boldsymbol{t} - x) - jg_2(\xi \cdot \boldsymbol{t} - x) - kg_3(\xi \cdot \boldsymbol{t} - x).$$

From equation (1.7) above, we will discuss the orthogonal properties which will be stated in the form of a theorem with the proof.

Theorem 3. *(ortogonal property). Let $\phi, \psi \in L^2(\mathbb{R}^2; \mathbb{H})$ and $f_1, f_2 \in L^2(\mathbb{R}^2; \mathbb{H})$, then the following orthogonality relation holds:*

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}} D_{s_\phi} f_1(\xi, x, \boldsymbol{\omega})\overline{D_{s_\psi} f_2(\xi, x, \boldsymbol{\omega})}\, dxd\boldsymbol{\omega} = (f_1(\bar{\phi}, \bar{\psi}), f_2). \tag{1.8}$$

Where

$$(\phi, \psi) = \int_{\mathbb{R}} \phi(\xi \cdot \boldsymbol{t} - x)\overline{\psi(\xi \cdot \boldsymbol{t} - x)}\, dx$$

Setting $u = \xi \cdot \boldsymbol{t} - x$

$$du = -dx$$

$$(\phi, \psi) = \int_{\mathbb{R}} \phi(u)\overline{\psi(u)}\, du$$

So that for $\phi = \psi$, then

$$\|\phi\|_{L^1(\mathbb{R};\mathbb{H})} = \int_{\mathbb{R}} |\phi(u)|^2\, du$$

Proof.

From **Theorem 2** , we have

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}} D_{s_\phi} f_1(\xi, x, \boldsymbol{\omega})\overline{D_{s_\psi} f_2(\xi, x, \boldsymbol{\omega})}\, dxd\boldsymbol{\omega}$$

$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} f_1(\boldsymbol{t})\overline{\phi(\xi \cdot \boldsymbol{t} - x)}\, e^{-2\pi i t_1 \omega_1} e^{-2\pi j t_2 \omega_2} d^2\boldsymbol{t}\right)$$

$$\times \overline{\left(\int_{\mathbb{R}^2} f_2(\boldsymbol{y})\,\overline{\psi(\xi \cdot \boldsymbol{y} - x)} e^{-2\pi i y_1 \omega_1}\, e^{-2\pi j y_2 \omega_2} d^2\boldsymbol{y}\right)} dxd\boldsymbol{\omega}$$

$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \Big(f_1(\boldsymbol{t})\overline{\phi(\xi \cdot \boldsymbol{t} - x)} e^{-2\pi i t_1 \omega_1} e^{-2\pi j t_2 \omega_2} e^{2\pi j y_2 \omega_2}\, e^{2\pi i y_1 \omega_1} \psi(\xi \cdot \boldsymbol{y}$$
$$- x)\, \overline{f_2(\boldsymbol{y})} d^2\boldsymbol{t}\, d^2\boldsymbol{y}\Big)\, dxd\boldsymbol{\omega}$$

$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(f_1(\boldsymbol{t})\overline{\phi(\xi\cdot\boldsymbol{t}-x)}e^{-2\pi i t_1\omega_1}e^{2\pi j\omega_2(y_2-t_2)}\,e^{2\pi i y_1\omega_1}\psi(\xi\cdot\boldsymbol{y} - x)\,\overline{f_2(\boldsymbol{y})}\,d^2\boldsymbol{t}\,d^2\boldsymbol{y}\right)dxd\boldsymbol{\omega}$$

By Fubini's theorem, we have

$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}} f_1(\boldsymbol{t})\overline{\phi(\xi\cdot\boldsymbol{t}-x)}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-2\pi i t_1\omega_1}e^{2\pi j\omega_2(y_2-t_2)}\,e^{2\pi i y_1\omega_1}\,d^2\boldsymbol{t}\,d^2\boldsymbol{y}\;\psi(\xi\cdot\boldsymbol{y} - x)\,\overline{f_2(\boldsymbol{y})}\,dxd\boldsymbol{\omega}$$

$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}} f_1(\boldsymbol{t})\overline{\phi(\xi\cdot\boldsymbol{t}-x)}\,\delta_1(y_2-t_2)\delta_2(y_1-t_1)\left(\psi(\xi\cdot\boldsymbol{y}-x)\;\overline{f_2(\boldsymbol{y})}\,d^2\boldsymbol{t}\,d^2\boldsymbol{y}\right)dx$$

We further get,

$$= \int_{\mathbb{R}^2} f_1(\boldsymbol{t})\left(\int_{\mathbb{R}}\overline{\phi(\xi\cdot\boldsymbol{t}-x)}\,\psi(\xi\cdot\boldsymbol{t}-x)\,dx\right)\overline{f_2(\boldsymbol{t})}\,d\boldsymbol{t}$$

$$= (f_1(\bar{\phi},\bar{\psi}), f_2).$$

So that

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}} D_{s_\phi}f_1(\xi,x,\boldsymbol{\omega})\overline{D_{s_\psi}f_2(\xi,x,\boldsymbol{\omega})}\,dxd\boldsymbol{\omega} = (f_1(\bar{\phi},\bar{\psi}), f_2),$$

which completes the proof.

Then, we obtain the following important consequences:

(i) If $\phi = \psi$, then

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}} D_{s_\phi}f_1(\xi,x,\boldsymbol{\omega})\overline{D_{s_\phi}f_2(\xi,x,\boldsymbol{\omega})}\,dxd\boldsymbol{\omega} = \|\phi\|_{L^1(\mathbb{R};\mathbb{H})}(f_1,f_2)$$

ii). If $f_1 = f_2$, then

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}} D_{s_\phi}f_1(\xi,x,\boldsymbol{\omega})\overline{D_{s_\psi}f_1(\xi,x,\boldsymbol{\omega})}\,dxd\boldsymbol{\omega} = (f_1(\bar{\phi},\bar{\psi}), f_1)$$

iii). If $f_1 = f_2$ dan $= \psi$, then

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}} \left|D_{s_\phi}f_1(\xi,x,\boldsymbol{\omega})\right|^2 dxd\boldsymbol{\omega} = \|\phi\|_{L^1(\mathbb{R};\mathbb{H})}\|f_1\|_2^2$$

■

CONCLUSIONS AND SUGGESTIONS

This paper has proven the orthogonal property of the directional short-time quaternion Fourier transform which is a generalization of the Directional short-time Fourier transform. The difference that can be seen in the orthogonal nature of the Directional short-time quaternion Fourier transform with the orthogonal properties of the Directional short-time Fourier transform is the notation used and of course. The resulting properties as well as the properties of the Short-Time Fourier Quaternion Transform has 3 important consequences that are not belongs to the short-time directional Fourier quaternion transform.

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