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MODULAR IRREGULAR LABELING ON COMPLETE GRAPHS

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Abstract
Let G be a simple graph of n order. An edge labeling such that the weights of all vertex are different and elements of the set modulo n, are called a modular irregular labeling. The modular irregularity strength of G is a minimum positive integer k such that G have a modular irregular labeling. If the modular irregularity strength is none, then we called the modular irregularity strength of G is infinity. In this article, we determine the modular irregularity strength of complete graphs.

Keywords: *Complete Graph; Irregular Labeling; Modular Irregular Labeling; Modular Irregularity Atrength*

INTRODUCTION

Graph labeling is a way of assigning labels in the form of positive integers to vertices, edges or both, with certain conditions. Until now, graph labeling has developed a lot, including irregular labeling and modular labeling. For other types of labeling, a reader can be read in A dynamic survey of graph labeling by Galian [5]. Furthermore, irreguler labeling described first by Chartrand et al. in 1986 [3]. Irregular labeling in graph G is defined as a mapping of the edge set $e \in E(G)$ to the set of integers $\{1,2,3,\dots,k\}$ such that all vertices $v \in V(G)$ have different weights. The weight of a vertex x is the sum of all the side labels associated with x, then the weight of the vertex x is usually denoted by $wt(x)$, where $wt(x) = \sum_{y \in N}(xy)$ with N is the set that is adjacent to x [4].

The irregularity strength of the graph G denoted by $s(G)$ is the minimum positive integer k such that graph G has k –irregular labeling. Irregularity strength is defined only for graphs with isolated vertices at most one and not connected to component with of order 2. Futhermore, $s(G) = \infty$ if no function satisfies. In other words, $s(G) < \infty$ if and only if, there is no isolated edges and at most only one isolated vertex [2]. Chartrand et al. in [4] give a lower bound on the graph as follows

$$s(G) \geq \max_{1 \leq i \leq \Delta} \left\{\frac{n_i + i - 1}{i}\right\}, \qquad (1)$$

where n_i is the number of vertices of degree i and Δ is the maximum degree of G.

In addition, Chartrand et al. in [4] also introduces regarding the lower bound on the strength irregularity in d –regular graph, where the lower bound for the strength irregularity of G, with $d \geq 2$ is

$$s(G) \geq \left\lceil \frac{n+d-1}{d} \right\rceil. \quad (2)$$

For the lower bound of the strength irregularity of r –regular graph, Faudree and Lehel [7,9] give results, if $r \geq 2$ and $n \geq 3$ then $s(G) \leq \left\lceil \frac{n}{2} \right\rceil + 9$ with the conjecture of $(G) \leq \left\lceil \frac{n}{r} \right\rceil + c$, c is a constant. In [7] also describes a proposition for every regular graph with order n, we get $s(G) \geq 3$. Chartrand et al. in [4] also prove that the lower bound for the strength irregularity in a complete graph, is $s(G) = 3$. Furthermore, for other strength irregularity can be checked in previous research. See [5] for the update.

In 2020, [2] introduced a labeling which is a modified result of irregular labeling. The result of these modifications is a modular irregular labeling. [2] also describes the modular irregular labeling of several types of graphs. The k –labeling edge $\psi : E(G) \rightarrow \{1,2,3, \dots, k\}$ is said modular irregular k –labeling on graph G if there is a bijective mapping on a defined weight function $\sigma : V(G) \rightarrow Z_n$ with

$$\sigma(x) = \sum_{y \in N(x)} \psi(xy), \quad (3)$$

where $N(x)$ is a point adjacent to x. It is said to be the modular weight of the vertex x, with Z_n is the set of integers modulo n and the sum of the labels of all the vertices adjacent to the vertex x. The minimum k of a graph G which is a k –labeling modular irregular is called the value of the modular irregularity of G denoted by $ms(G)$. If there is no modular labeling for the graph G, then the value of $ms(G)$ is defined as $ms(G) = \infty$ [2].

It is known that, [2] gives the lower bound of the value of the modular irregularity if G is a graph with no component of order ≤ 2, then

$$s(G) \leq ms(G). \quad (4)$$

In [2] presents several theorems that are used to give the modular irregularity strength of a graph.

Theorem 1 (see [2]). *If G is a graph of order n, $n \equiv 2(mod 4)$ then G has no modular irregular k –labeling i.e., $ms(G) = \infty$.*

Chartrand et al. [4] has discussed the lower bound of the strength irregularity in a reguler graph which is explained in the following theorem.

Theorem 2 (see [4]). *If G is a r –reguler graph of order p, then*

$$s(G) \geq \frac{p-1}{r} + 1. \quad (5)$$

Following this theorem, [4] gives a proposition about the lower bound of the strength irregularity in a complete graph.

Proposition 1 (see [4]). *For each $n \geq 3$,*

$$s(K_n) = 3. \quad (6)$$

Baca et al. [2] sets the lower bound for the value of modular irregularity and determines the modular irregularity of 5 types of graphs, namely path, star, triangular graph, cycle, and gear. Next, Baca et al. [10] provide the irregularity value of the fan graph. Muthugurupackiam et al. [11] provide the value of the irregularity of two classes of graphs, specifically tadpole graphs and double-cycle graphs. In [12], Baca et al. determine the value of the irregularity of the wheel graph. Latest, in 2021, [8] Sugeng et al. determines the irregularity value of 2 types of graphs, namely a regular double star graph and a friendly graph. In this research, we will examine the value of the modular irregularity of the complete graph K_n.

In this research, we will examine the value of the modular irregularity of the complete graph K_n. In determining the modular irregularity strength in a complete graph, this paper is structured in several stages as follows. First, we define a complete graph as the graph to be used. Furthermore, based on this definition, a labeling function is constructed in the form of an algorithm to obtain the modular irregularity strength. The algorithm is divided into 5 cases, where in each case we will get the largest k. After that, the weight of each vertex in the graph is determined. Last, we will show that the weight of each vertex is different and is an element of Z_n.

RESULT AND DISCUSSION

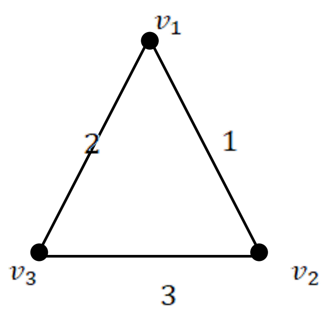
In this section, we will describe the results of modular irregular labeling on a complete graph K_n. In addition, it will also show the lower bound of the strength irregularity in the complete graph according to Proportion 1 using the algorithm that has been made. A complete graph is a graph consisting of vertices set $\{v_1, v_2, \dots, v_n\}$ and edge set $\{v_i v_j \mid v_i, v_j \in V, i \neq j\}$. So, a complete graph has n vertices and $\frac{n(n-1)}{2}$ edges.

Irregularity Strength of Complete Graph

In this section, we will discuss the irregularity strength in complete graphs. In [4] it is said that the lower bound of the irregularity strength of a complete graph is $s(K_n) = 3$. To show whether 3 is also the upper bound of the complete graphs irregularity strength, then labeling function construction is carried out in the form of an algorithm to show the upper bound. Where the algorithm will divide into two cases, when $n = 3$ and $n \geq 4$.

Case 1. $n = 3$
To determine the weight vertex irregularity of the complete graph K_3, it can be seen the k –labeling of the edge irregularity of the complete graph K_3 as shown in Figure 1.

Figure 1. A complete graph K_3



In Figure 1, it can be seen that the labeling on the complete graph $K_3$is an irregular-3 labeling. The weights for each node are shown in Table 1.

Tabel 1. Matrix of irregularity strength of the complete graph K_3

$\varphi(v_i)$	v_1	v_2	v_3	Weight
v_1	0	1	2	3
v_2	1	0	3	4
v_3	2	3	0	5

In the last column of Table 1, it can be seen that each row has different entries. This shows that the weight of each vertex of the complete graph K_3 is different. Since the largest entry used in Table 1 is 3, then $s(K_3) \leq 3$.

Case 2. $n \geq 4$

Algorithm 1.

1. Build an $n \times n$ symmetric matrix, with the $-i$th row and $-j$th column of this matrix corresponding to the vertex v_i dan v_j in a complete graph K_n.
2. The entries of the $-i$th row and $-j$th column of this matrix correspond to the edge labels $v_i v_j$ in the complete graph K_n.
3. The main diagonal entries are labeled 0.
4. The entries of the $-i$th row and $-j$th column are labeled 1, if

   $i = 1,\ j = 2,3, \dots, n$

   $i = 2,\ j = 3,4, \dots, n-1$

   .

   .

   .

   $i = \left\lfloor \frac{n}{2} \right\rfloor,\ j = i+1, i+2, \dots, n-i+1$

   so, label 1 starts from column $1,2, \dots, n-i+1$.
5. The entries of the $-i$th row and $-j$th column are labeled in 2, if

   $i = 2,\ j = n$

   $i = 3,\ j = n-1, n$

   .

   .

   .

   $$i = n-2, j = \begin{cases} n-(i-2), n-(i-3), \dots, n, & i \geq \dfrac{n+2}{2} \\ n-(i-3), n-(i-4), \dots, n-1, & i < \dfrac{n+2}{2} \end{cases}$$

   so, label 2 starts from column $n-i+2, n-i+3, \dots, n-i+\left\lceil \frac{n}{2} \right\rceil$.
6. The entries of the $-i$th row and $-j$th column are labeled in 3, if

   $i = n - \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right),\ j = n$

   .

   .

   .

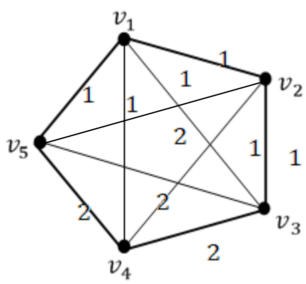
   $i = n-1,\ j = n$

   so, label 3 starts from column $n-i+\left\lceil \frac{n}{2} \right\rceil + 1, n-i+\left\lceil \frac{n}{2} \right\rceil + 2, \dots, n$
7. Count all column entries in a given row, using

$$\sigma(i) = \sum_{i=1}^{n} \varphi(i, j),$$

thus, the weight of each vertex is different.

Figure 2. A complete graph K_5



Use Figure 2 and following an algorithm based on each case criteria, a matrix as shown in Table 2.

Tabel 2. Matrix of irregularity strength of the complete graph K_5

$\varphi(v_i)$	v_1	v_2	v_3	v_4	v_5	**Weight**
v_1	0	1	1	1	1	4
v_2	1	0	1	1	2	5
v_3	1	1	0	2	2	6
v_4	1	1	2	0	3	7
v_5	1	2	2	3	0	8

It is clear, that in the algorithm and Table 2, the largest obtained is 3, with each vertex weight being different. The research results obtained using Algorithm 1 are as follows.

Theorem 3. *For every $n \geq 3$, strength irregularity of the complete graph is*

$$s(K_n) = 3$$

Proof.

a. It will be proved that $s(K_n) \geq 3$.
   Complete graph is a reguler graph. From Theorem 2 and Proposition 3, we known that every reguler graph G with order n, $n \geq 3$, then $s(G) \geq 3$. Furthermore, complete graph K_n with order n, $n \geq 3$, can be proved that $s(K_n) \geq 3$.
b. It will be proved that $s(K_n) \leq 3$
   In proving $s(K_n) \leq 3$, it is necessary to construct an irregular k –labeling of vertices as follows. Based on Algorithm 1, a labeling function is constructed as follows.
   For $i = 1,2, \dots, n-1, i \neq j$, then

$$\varphi(v_i v_j) = \begin{cases} 1, j = 1,2, \dots, n-i+1 \\ 2, j = n-i+2 \\ 3, j = n-i+3, n-1+4, \dots, n \end{cases} \tag{7}$$

   and, for $i = n$, then

$$\varphi(v_n v_j) = \begin{cases} 1,\ j = 1,2, \dots, n-i+1 \\ 2,\ j = n-i+2, n-i+3, \dots, n-i+\left\lceil\frac{n}{2}\right\rceil \\ 3,\ j = n-i+\left\lceil\frac{n}{2}\right\rceil+1, n-i+\left\lceil\frac{n}{2}\right\rceil+2, \dots, n \end{cases} \tag{8}$$

Based on the constructed function, it is found that the largest label used is 3. Furthermore, it will be proven that each node has a different weight. For that purpose, we will add up for each row entry which is the weight of the node corresponding to a particular row, is

$$\sigma(v_i) = \sum_{j=1}^{n} \varphi(v_i v_j).$$

Thus, it is obtained that for a vertex $v_i, i = 1,2,\dots,n$, the weight of vertex v_i is

a. For $i = 1,2,\dots,n-1, i \neq j$, then

$$\sigma(v_i) = \varphi(v_i v_i) + \sum_{j=1}^{n-i+1} \varphi(v_i v_j) + \sum_{j=n-i+2}^{n-i+2} \varphi(v_i v_j) + \sum_{j=n-i+3}^{n} \varphi(v_i v_j)$$
$$= 0 + 1(n-i+1) + 1 + 2((n-i+2) - (n-i+2) + 1) + 3(n - (n-i+3+1)) - 1$$
$$= n - i + 2 + 2(1) + 3(i-2) - i$$
$$= n + i - 2 \qquad (9)$$

b. For $i = n$, then

$$\sigma(v_n) = \varphi(v_n v_n) + \sum_{j=1}^{1} \varphi(v_n v_j) + \sum_{j=2}^{\lceil \frac{n}{2} \rceil} \varphi(v_n v_j) + \sum_{j=\lceil \frac{n}{2} \rceil + 1}^{n} \varphi(v_n v_j)$$
$$= 0 + 1(1) + 2\left(\left\lceil \frac{n}{2} \right\rceil - 2 + 1\right) + 3\left(n - \left\lceil \frac{n}{2} \right\rceil + 1\right)$$
$$= 1 + 2\left\lceil \frac{n}{2} \right\rceil - 2 + 3n - 3\left\lceil \frac{n}{2} \right\rceil - 3$$
$$= 3n - \left\lceil \frac{n}{2} \right\rceil - 4 \qquad (10)$$

Based on (9) and (10) it is obtained that each vertex has a different weight. Thus, the labeling function is

$$\varphi: E \to \{1,2,\dots,k\}$$

define an irregular-3 labeling for complete graph $K_n, n \geq 4$. Then, $s(K_n) \leq 3$.

So, it is obtained that $s(K_n) = 3$. ■

Modular Irregular Labeling Algorithm of Complete Graph

In this section, we will discuss the algorithm for determining the modular irregularity strength in a complete graph. To obtain these results, labels are carried out on the edges of the graph by looking at the following things.

1. Determine the lower bound according to Section "Irregularity Strength of Complete Graph".
2. Determine the upper bound by giving a label and constructing a label on the edges of graph using the matrix, so that the largest label edge k is obtained.
3. Calculate the weight of vertex, with the weight of each vertex is different.

From Section "Irregularity Strength of Complete Graph", we get $s(K_n) = 3$. So, for the next step, we need to find the upper bound of the modular irregularity strength. To get the upper bound, an algorithm is formed with several cases that have been divided by a certain n.

Matrix Algorithm to Calculate $ms(K_n)$ with $n = 3$

Using the labeling in Case 1, it is obtained that $s(K_3) = 3$. Following the labeling, it is obtained that $ms(K_3) = s(K_3)$, where the weight interval in modulo 3 $v_i = i - 1$. By using the explanations, the matrix corresponding to the complete graph K_3 is obtained as follows in Table 3.

Tabel 3. Matrix of modular irregularity strength of the complete graph K_3

$\varphi(v_i)$	v_1	v_2	v_3	Weight
v_1	0	1	2	0
v_2	1	0	3	1
v_3	2	3	0	2

In the last column of Table 3, it can be seen that each row has different entries and is an element of the set $\mathbb{Z}_3$. This shows that the weight of each vertex of the complete graph K_3 is different. Since the largest entry used in Table 3 is 3, then $ms(K_3) \leq 3$.

Matrix Algorithm to Calculate $ms(K_n)$ with $n = \{4, 5\}$

In determining the matrix entries in the complete graph $K_n, n = \{4,5\}$, an algorithm is formed as in Algorithm 1. By using Algorithm 1, the matrix corresponding to the complete graph K_5 is obtained as follows in Table 4.

Tabel 4. Matrix of modular irregularity strength of the complete graph K_5

$\varphi(v_i)$	v_1	v_2	v_3	v_4	v_5	Weight
v_1	0	1	1	1	1	4
v_2	1	0	1	1	2	0
v_3	1	1	0	2	2	1
v_4	1	1	2	0	3	2
v_5	1	2	2	3	0	3

Matrix Algorithm to Calculate $ms(K_n)$ with $n \equiv 1, 3 \ (mod\ 4), n \geq 7$

By using Algorithm 1, the weight of each vertex can be determined by adding several steps to the algorithm that has been made as follows:
Algorithm 2.

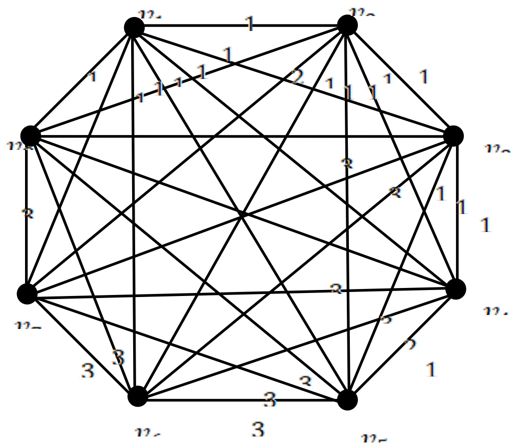
1. Change entries for rows and columns $(3, n)$ with label 3.
2. Change the entries in the matrix diagonally up and the rows below the diagonal starting from the entry $(n, 3)$ up to column $\left\lfloor \frac{n}{2} \right\rfloor + 2$ with label 3.
3. Fill in the column entries $\left\lfloor \frac{n}{2} \right\rfloor + 1$ according to the previous column entries.
4. Change the entries in the matrix diagonally downwards and the rows below the diagonal starting from column $\left\lfloor \frac{n}{2} \right\rfloor + 3$ to matrix entries $(n, n-1)$ with label 3.
5. Count all column entries in a given row, using

$$\sigma(i) = \sum_{j=1}^{n} \varphi(i, j)$$

Matrix Algorithm to Calculate $ms(K_n)$ with $n = 8$

To determine the modular strength irregularity of the complete graph K_8, it can be seen the k –labeling of the edge irregularity of the complete graph K_8 as shown in Figure 3.

Figure 3. A complete graph K_8



In Figure 3, it can be seen that the labeling on the complete graph $K_8$is a modular irregular-3 labeling. The weights for each vertex are shown in Table 5.

Tabel 5. Matrix of modular irregularity strength of the complete graph K_5

$\varphi(v_i)$	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	Weight
v_1	0	1	1	1	1	1	1	1	7
v_2	1	0	1	1	1	1	1	2	0
v_3	1	1	0	1	1	1	3	3	3
v_4	1	1	1	0	1	2	3	3	4
v_5	1	1	1	1	0	3	3	3	5
v_6	1	1	1	2	3	0	3	3	6
v_7	1	1	3	3	3	3	0	3	1
v_8	1	2	3	3	3	3	3	0	2

In the last column of Table 5, it can be seen that each row has different entries and is an element of the set $\mathbb{Z}_8$. This shows that the weight of each vertex of the complete graph K_8 is different. Since the largest entry used in Table 3 is 3, then $ms(K_8) \leq 3$.

Matrix Algorithm to Calculate $ms(K_n)$ with $n \equiv 0\ (mod\ 4), n \geq 12$

By using Algorithm 1, the weight of each vertex can be determined by adding several steps to the algorithm that has been made as follows:
Algorithm 3.

1. The number of rows with entries labeled 3 in each column is i starting from column 3, where $i = \left\{1,2,3, \ldots, \frac{n}{2} - 1\right\}, i \neq \frac{n}{4} - 1$.

2. Change the $\frac{n}{2}$ column entries with label 3 on the row to match the number of rows (1).
3. Column entry $\frac{n}{2}-1$ matches the column entry after it.
4. Change the entries in the column diagonally up and the row below the diagonal starting from column $\frac{n}{2}-1$ to column $\frac{n}{2}+1$ with label 3.
5. Change the entries in the matrix diagonally downwards and the rows below the diagonal starting from column $\left\lfloor\frac{n}{2}\right\rfloor+3$ to matrix entries $(n, n-1)$ with label 3.
6. Count all column entries in a given row, using

$$\sigma(i)=\sum_{j=1}^{n}\varphi(i,j)$$

Modular Irregularity Strength of Complete Graph

In this section, we will discuss the results obtained in the previous section and will discuss several previous research. The results obtained in this section will then be used as conclusions.

Theorem 4. *For every complete graph* $K_n, n \geq 3$ *and* $n \equiv 0,1,3\ (mod\ 4)$ *Then modular irregularity strength is*

$$ms(K_n)=3$$

Proof.

a. It will be proved that $ms(K_n) \geq 3$.
   In [2], Baca et al. gives the lower bound of the value of the modular irregularity if G is a graph with no component of order ≤ 2, then $ms(G) \geq s(G)$, so it can be said that $ms(K_n) \geq s(K_n)$ We know that $s(K_n)=3$, so we obtained that $ms(K_n) \geq 3$.
b. It will be proved that $ms(K_n) \leq 3$.
   We will devide the proof in 5 cases.

Case 1. For $n=3$

For the complete graph K_3, a modular irregular k –labeling is given in section Matrix Algorithm to Calculate $ms(K_n)$ with $n=3$. Then, it is obtained that $ms(K_n) \leq 3$.

Case 2. For $n=4,5$

For the complete graph K_4 and K_5, a modular irregular k –labeling is given as follows. Based on Algorithm 1, a labeling function is constructed as follows.

For $i=1,2,\dots,n, i \neq j$, then

$$(v_i v_j)=\begin{cases}1, & j=1,2,\dots,n-i+1\\ 2, & j=n-i+2\\ 3, & j=n-i+3, n-1+4,\dots,n\end{cases} \tag{11}$$

Based on the constructed function, it is found that the largest label used is 3. Furthermore, it will be proven that each vertex has a different weight. For that purpose, we will add up for each row entry which is the weight of the vertex corresponding to a particular row, is

$$\sigma(v_i) = \sum_{j=1}^{n} \varphi(v_i v_j).$$

Thus, it is obtained that for a vertex $v_i, i = 1,2, \dots, n$, the weight of vertex v_i is,

For $i = 1,2, \dots, n, i \neq j$, then

$$\begin{aligned}\sigma(v_i) &= \varphi(v_i v_i) + \textstyle\sum_{j=1}^{n-i+1} \varphi(v_i v_j) + \sum_{j=n-i+2}^{n-i+2} \varphi(v_i v_j) + \sum_{j=n-i+3}^{n} \varphi(v_i v_j) \\ &= 0 + 1(n-i+1) + 1 + 2((n-i+2) - (n-i+2) + 1) + 3(n - (n-i + \\ &\quad 3+1)) - 1 \\ &= n - i + 2 + 2(1) + 3(i-2) - i \\ &= n + i - 2 \end{aligned} \tag{12}$$

Based on (12), then the weight interval in modulo n is $i - 2$. Therefore, it is obtained that each vertex has a different weight and is an element of the set $\mathbb{Z}_n, n = \{4,5\}$. Thus, the labeling function section,

$$\varphi: E \to \{1,2, \dots, k\}$$

define an irregular-3 labeling for complete graph $K_n, n = 4,5$ Then, $ms(K_n) \leq 3$.

Case 3. For $n \equiv 1,3 \ (mod\ 4), n \geq 7$

For the complete graph K_n, $n \equiv 1,3 \ (mod\ 4)$, $n \geq 7$, a modular irregular k –labeling is given as follows. Based on Algorithm 2, a labeling function is constructed as follows.

For $i = 1$, then

$$(v_1 v_j) = 1, j = 2, \dots, n \tag{13}$$

For $\left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 3$, then

$$(v_i v_j) = \begin{cases} 1, \ j = 1,2, \dots, n-i+1 \\ 2, \ j = n-i+2, n-i+3 \\ 3, \ j = n-i+4, n-1+5, \dots, n \end{cases} \tag{14}$$

For other i, then

$$\varphi(v_i v_j) = \begin{cases} 1, \ j = 1,2, \dots, n-i+1 \\ 2, \ j = n-i+2 \\ 3, \ j = n-i+3, n-1+4, \dots, n \end{cases} \tag{15}$$

Based on the constructed function, it is found that the largest label used is 3. Furthermore, it will be proven that each vertex has a different weight. For that purpose, we will add up for each row entry which is the weight of the vertex corresponding to a particular row, is

$$\sigma(v_i) = \sum_{j=1}^{n} \varphi(v_i v_j)$$

Thus, it is obtained that for a vertex $v_i, i = 1,2, \dots, n$, the weight of vertex v_i is,

1. For $i = 1$, label 1 is given starting from column $2,3 \dots, n$, then there is no label 2 and label 3. So we get
$$\sigma(i) = n - 1. \tag{16}$$
2. For $2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$, label 1 is given starting from column $1,2, \dots, n-i+1$, label 2 assigned to column $n-i+2$ and label 3 starting from column $n-i+3, n-i+4, \dots, n$. So we get
$$\sigma(i) = n + 2i - 4. \tag{17}$$

3. For $\left\lfloor\frac{n}{2}\right\rfloor+1 \leq i \leq \left\lfloor\frac{n}{2}\right\rfloor+3$, label 1 is given starting from column $1,2,\dots,n-i+1$, label 2 assigned to column $(n-i+2, n-i+3)$ and label 3 starting from column $n-i+4, n-i+5,\dots,n$. So we get
$$\sigma(i) = n + i + \left\lfloor\frac{n}{2}\right\rfloor - 4. \tag{18}$$
4. For $i \geq \left\lfloor\frac{n}{2}\right\rfloor+4$, label 1 is given starting from column $1,2,\dots,n-i+1$, label 2 assigned to column $n-i+2$ and label 3 starting from column $n-i+3, n-i+4,\dots,n$. So we get
$$\sigma(i) = n + 2i - 6. \tag{19}$$

Thus, it can be seen that the weight interval in modulo n is as follows
1. For $i=1$, the vertex weight v_i is $n-1$.
2. For $2 \leq i \leq \left\lfloor\frac{n}{2}\right\rfloor$, the vertex weight interval v_i is even interval $[0, n-5]$.
3. For $\left\lfloor\frac{n}{2}\right\rfloor+1 \leq i \leq \left\lfloor\frac{n}{2}\right\rfloor+3$, the vertex weight inteval v_i is $[n-4, n-2]$.
4. For $i \geq \left\lfloor\frac{n}{2}\right\rfloor+4$, the vertex weight interval v_i is odd interval $[1, n-6]$.

Therefore, it is obtained that each vertex has a different weight and is an element of the set $\mathbb{Z}_n, n \equiv 1,3 \ (mod\ 4)$, $n \geq 7$. Thus, the labeling function section,
$$\varphi: E \to \{1,2,\dots,k\}$$
define an irregular-3 labeling for complete graph $K_n, n \equiv 1,3\ (mod\ 4)$, $n \geq 7$. Then, $ms(K_n) \leq 3$.

Case 4. For $n=8$

For the complete graph K_8, a modular irregular k –labeling is given in section Matrix Algorithm to Calculate $ms(K_n)$ with $n=8$. Then, it is obtained that $ms(K_8) \leq 3$.

Case 5. For $n \equiv 0\ (mod\ 4), n \geq 12$

For the complete graph K_n, $n \equiv 0\ (mod\ 4)$, $n \geq 12$, a modular irregular k –labeling is given as follows. Based on Algorithm 3, a labeling function is constructed as follows.
For $i=1$, then
$$(v_1 v_j) = 1, j = 2,\dots,n \tag{20}$$

For $\frac{n}{4}+1 \leq i \leq \frac{2n}{4}-2$, then
$$\varphi(v_i v_j) = \begin{cases} 1, & j = 1,2,\dots,n-i+1 \\ 3, & j = n-i+2, n-i+3,\dots,n \end{cases} \tag{21}$$

For $\frac{2n}{4}+3 \leq i \leq \frac{3n}{4}$, then
$$\varphi(v_i v_j) = \begin{cases} 1, & j = 1,2,\dots,n-i+1 \\ 3, & j = n-i+2, n-i+3,\dots,n \end{cases} \tag{22}$$

For other i, then
$$\varphi(v_i v_j) = \begin{cases} 1, & j = 1,2,\dots,n-i+1 \\ 2, & j = n-i+2 \\ 3, & j = n-i+3, n-1+4,\dots,n \end{cases} \tag{23}$$

Based on the constructed function, it is found that the largest label used is 3. Furthermore, it will be proven that each vertex has a different weight. For that purpose, we will add up for each row entry which is the weight of the vertex corresponding to a particular row, is

$$\sigma(v_i) = \sum_{j=1}^{n} \varphi(v_i v_j)$$

Thus, it is obtained that for a vertex $v_i, i = 1,2, \dots, n$, the weight of vertex v_i is,

1. For $i = 1$, label 1 is given starting from column $2,3 \dots, n$, then there is no label 2 and label 3. So we get
$$\sigma(i) = n - 1. \tag{24}$$
2. For $2 \le i \le \frac{n}{4}$, label 1 is given starting from column $1,2, \dots, n - i + 1$, label 2 assigned to column $n - i + 2$ and label 3 starting from column $n - i + 3, n - i + 4, \dots, n$. So we get
$$\sigma(i) = n + 2i - 4. \tag{25}$$
3. For $\frac{n}{4} + 1 \le i \le \frac{2n}{4} - 2$, label 1 is given starting from column $1,2, \dots, n - i + 1$, no label 2 and label 3 starting from column $n - i + 2, n - i + 3, \dots, n$. So we get
$$\sigma(i) = n + 2i - 3. \tag{26}$$
4. For $\frac{2n}{4} - 1 \le i \le \frac{2n}{4} + 2$, label 1 is given starting from column $1,2, \dots, n - i + 1$, label 2 assigned to column $n - i + 2$ and label 3 starting from column $n - i + 3, n - i + 4, \dots, n$. So we get
$$\sigma(i) = \frac{3n}{2} + i - 4. \tag{27}$$
5. For $\frac{2n}{4} + 3 \le i \le \frac{3n}{4}$, label 1 is given starting from column $1,2, \dots, n - i + 1$, no label 2 and label 3 starting from column $n - i + 2, n - i + 3, \dots, n$. So we get
$$\sigma(i) = n + 2i - 5. \tag{28}$$
6. For $\frac{3n}{4} + 1 \le i \le \frac{3n}{4} + 2$, label 1 is given starting from column $1,2, \dots, n - i + 1$, label 2 assigned to column $n - i + 2$ and label 3 starting from column $n - i + 3, n - i + 4, \dots, n$. So we get
$$\sigma(i) = \frac{7n}{4} + i - 4. \tag{29}$$
7. For $\frac{3n}{4} + 3 \le i \le n$, label 1 is given starting from column $1,2, \dots, n - i + 1$, label 2 assigned to column $n - i + 2$ and label 3 starting from column $n - i + 3, n - i + 4, \dots, n$. So we get
$$\sigma(i) = n + 2i - 6 \qquad (29)$$

Thus, it can be seen that the weight interval in modulo n is as follows

1. For $i = 1$, the vertex weight v_i is $n - 1$.
2. For $2 \le i \le \frac{n}{4}$, the vertex weight interval v_i is even interval $[0, n - 8]$.
3. For $\frac{n}{4} + 1 \le i \le \frac{2n}{4} - 2$, the vertex weight inteval v_i is is odd interval $\left[\frac{n}{2} - 1, n - 7\right]$.
4. For $\frac{2n}{4} - 1 \le i \le \frac{2n}{4} + 2$, the vertex weight interval v_i is $[n - 5, n - 2]$.
5. For $\frac{2n}{4} + 3 \le i \le \frac{3n}{4}$, the vertex weight inteval v_i is is odd interval $\left[1, \frac{n}{2} - 5\right]$.
6. For $\frac{3n}{4} + 1 \le i \le \frac{3n}{4} + 2$, the vertex weight interval v_i is $\left[\frac{n}{2} - 3, \frac{n}{2} - 2\right]$.

7. For $\frac{3n}{4}+3 \leq i \leq n$, the vertex weight interval v_i is even interval $\left[\frac{n}{2}, n-6\right]$.

Therefore, it is obtained that each vertex has a different weight and is an element of the set $\mathbb{Z}_n, n \equiv 0 \ (mod\ 4)$, $n \geq 12$. Thus, the labeling function section,

$$\varphi: E \rightarrow \{1,2,\ldots,k\}$$

define an irregular-3 labeling for complete graph $K_n, n \equiv 1,3 \ (mod\ 4)$, $n \geq 7$. Then, $ms(K_n) \leq 3$. Based on 5 cases, it is obtained that each vertex has a different weight. Thus, the labeling function is

$$\varphi: E \rightarrow \{1,2,\ldots,k\}$$

define an irregular-3 labeling for complete graph $K_n, n \equiv 0,1,3 \ (mod\ 4), n \geq 3$. Then, m$s(K_n) \leq 3$.

So, it is obtained that $ms(K_n) = 3$. ■

CONCLUSIONS AND SUGGESTIONS

In this paper, we determine the exact value of the irregularity strength of a complete graph of order n, $n \geq 3$, as follows

$$s(K_n) = 3$$

From the irregular assignment of the complete graph, modifications were made to obtain a modular irregular labeling. The assignment returns the exact value of modular irregularity strength of complete graph K_n.

$$ms(K_n) = \begin{cases} 3, & n \not\equiv 2 \ mod\ 4 \\ \infty, & n \equiv 2 \ mod\ 4 \end{cases}$$

Based on the results obtained, it is known that a complete graph is a graph with the same irregularity strength as the modular irregularity strength, except in the case of $n \equiv 2 \ mod\ 4$ for the modular irregularity strength as discussed in Theorem 1. For further research, we suggest the following open problems.

Problem 1. *Verify if there are other families of regular graphs whose irregular strength and modular ir-regular strength are the same.*

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